University of Illinois Fall 2012

ECE 313: Hour Exam II

Monday November 12, 2012
7:00 p.m. — 8:15 p.m.
124 Burill Hall (Section X) & 100 Noyes Lab (Sections C,D,E)

1. (a) The number of tweets in one week is a Poisson random variable with parameter $(1/7)7 = 1$. Let W_i be the event you receive exactly one tweet on week i. Then $P(W_i) = e^{-1}1^1/1! = e^{-1}$, and by the independence of non-overlapping intervals, $P(W_1, W_2, W_3, W_4) = P(W_1)P(W_2)P(W_3)P(W_4) = \left(e^{-1}\right)^4 = e^{-4}$.

   (b) The time between consecutive tweets has the exponential distribution with parameter 1/7, and thus mean 7 days. By the memoryless property of the exponential distribution, the mean amount of additional time until the arrival of the fifth tweet is $7 * 5 = 35$ days, or five weeks.

2. (a) If $S = 55$, there are 55 heads and 45 tails, and the difference is 10. So if $S \geq 55$, the difference is greater than or equal to ten. Similarly, if $S \leq 45$, the difference is less than or equal to -10. So $A = \{S \geq 55 \text{ or } S \leq 45\}$, or equivalently, $A = \{|S - 50| \geq 5\}$. ALTERNATIVELY, the number of heads minus the number of tails is $S - (100 - S) = 2S - 100S$. So $A = \{|2S - 100| \geq 10\} = \{|S - 50| \geq 5\}$.)

   (b) The random variable S has the binomial distribution with parameters $n = 100$ and $p = 0.5$. By the symmetry of the distribution of S about 50, $P\{S \leq 45\} = P\{S \geq 55\}$, so $P(A) = 2P\{S \geq 55\}$. Since S takes integer values with $E[S] = 50$, $\mathrm{Var}(S) = (100)p(1 - p) = 25$, and standard deviation $\sigma = \sqrt{25} = 5$,

$$P(A) = 2P\{S \geq 54.5\} = 2P\left\{\frac{S - 50}{5} \geq \frac{54.5 - 50}{5}\right\} \approx 2Q\left(\frac{54.5 - 50}{5}\right) = 2Q(0.9).$$

3. The density is symmetric about a so the mean is a. Therefore $a = \mu_X = 2$.

   In order for f_X to be a valid pdf, it must integrate to one. The region under f_X consists of a rectangle of area $2b(1)$ and two triangles of area $c(1)/2$ each, so its total area is $2b + c$; therefore, $2b + c = 1$.

   By inspection of the figure, $F_X(a - b)$ is the area of the first triangle, so $F_X(a - b) = \frac{c}{2}$. By the formula given for F_X, $F_X(a - b) = \frac{5c^2}{6}$. So $\frac{c}{2} = \frac{5c^2}{6}$, or $c = 0.6$. (ALTERNATIVELY, taking the derivative of the expression given form F_X shows that $f_X(u) = \frac{5}{3}(u - (a - b - c))$ in the interval $[a - b - c, a - b]$. Setting $f_X(a - b) = 1$ givens $\frac{5c}{3} = 1$ or $c = 0.6$. ALTERNATIVELY, taking the derivative of F_X twice yields $F_X'' = f_X' = \frac{5}{3}$ over the interval $[a - b - c, a - b]$. By the picture, the slope of f_X over the interval is $\frac{1}{c}$. So $\frac{5}{3} = \frac{1}{c}$ or $c = 0.6$. There are other ways to derive $c = 0.6$; they all involve comparing the picture of f_X over the interval $[a - b - c, a - b]$ to the formula given for F_X.) Since $2b + c = 1$, $b = 0.2$.

4. (a)

$$F_T(t) = 1 - e^{-\int_0^t h(s)ds} = \begin{cases} 1 - e^{-t} & \text{if } 0 < t < 1 \\ 1 - e^{1-2t} & \text{if } t \geq 1 \end{cases}$$

   (b) No. For example, $P\{T > 1\} = 1 - F_T(1) = e^{-1}$, whereas

$$P(T > 2|T > 1) = \frac{P(T > 2, T > 1)}{P\{T > 1\}} = \frac{P\{T > 2\}}{P\{T > 1\}} = \frac{1 - F_T(2)}{1 - F_T(1)} = \frac{e^{-3}}{e^{-1}} = e^{-2}$$

5. (a) For $-1 \leq u \leq 1$ the likelihood ratio is given by $\Lambda(u) = 2|u|$. The MAP rule declares H_1 if $\Lambda(u) \geq \frac{\pi_0}{\pi_1} = \frac{1}{2}$, i.e., if $|u| \geq \frac{1}{4}$, and declares H_0 if $|u| < \frac{1}{4}$.

(b)

$$p_{\text{false alarm}} = \int_{\frac{1}{4} \leq |u| \leq 1} f_0(u)du = \int_{\frac{1}{4} \leq |u| \leq 1} \frac{1}{2}du = 2\int_{\frac{1}{4} \leq u \leq 1} \frac{1}{2}du = \frac{3}{4}$$

6. (a) No, because, for example, the support is not a product set.

(b)

$$f_X(u) = \begin{cases} \int_{u^2}^{1} \frac{3}{2}\, dv \;=\; \frac{3}{2}(1-u^2), & 0 \leq u \leq 1 \\ 0, & u < 0 \text{ or } u > 1 \end{cases}$$

(c) For $0 \leq u < 1$. (We will not worry in grading whether the values $u = 0$ or $u = 1$ are included or not.)

(d) For $0 \leq u < 1$,

$$f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(u)} = \begin{cases} \frac{1}{1-u^2}, & u^2 \leq v \leq 1 \\ 0, & v < u^2 \text{ or } v > 1 \end{cases}$$

That is, for $0 \leq u < 1$, given $X = u$, Y is uniformly distributed over the interval $[u^2, 1]$.

(e)

$$P\{Y > X\} = \int_0^1 \left[\int_u^1 \frac{3}{2}\, dv\right] du = \frac{3}{4}$$

or simply $\frac{3}{2}\times\text{Area} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$