1. (a) The number of tweets in one week is a Poisson random variable with parameter \((1/7)7 = 1\). Let \(W_i\) be the event you receive exactly one tweet on week \(i\). Then \(P(W_i) = e^{-1 \frac{1}{7}} / 1! = e^{-1}\), and by the independence of non-overlapping intervals, \(P(W_1, W_2, W_3, W_4) = P(W_1)P(W_2)P(W_3)P(W_4) = (e^{-1})^4 = e^{-4}\).

(b) The time between consecutive tweets has the exponential distribution with parameter \(1/7\), and thus mean 7 days. By the memoryless property of the exponential distribution, the mean amount of additional time until the arrival of the fifth tweet is \(7\) days.

2. (a) If \(S = 55\), there are 55 heads and 45 tails, and the difference is 10. So if \(S \geq 55\), the difference is greater than or equal to ten. Similarly, if \(S \leq 45\), the difference is less than or equal to -10. So \(A = \{S \geq 55 \text{ or } S \leq 45\}\), or equivalently, \(A = \{|S-50| \geq 5\}\). ALTERNATIVELY, the number of heads minus the number of tails is \(S - (100 - S) = 2S - 100\). So \(A = \{2S - 100 \geq 10\} = \{|S-50| \geq 5\}\).

(b) The random variable \(S\) has the binomial distribution with parameters \(n = 100\) and \(p = 0.5\). By the symmetry of the distribution of \(S\) about 50, \(P(S \leq 45) = P(S \geq 55)\), so \(P(A) = 2P(S \geq 55)\). Since \(S\) takes integer values with \(E[S] = 50\), \(Var(S) = (100)p(1-p) = 25\), and standard deviation \(\sigma = \sqrt{25} = 5\).

\[
P(A) = 2P\{S \geq 54.5\} = 2P\left\{ \frac{S - 50}{5} \geq \frac{54.5 - 50}{5} \right\} \approx 2Q\left( \frac{54.5 - 50}{5} \right) = 2Q(0.9).
\]

3. The density is symmetric about \(a\) so the mean is \(a\). Therefore \(a = \mu_X = 2\).

In order for \(f_X\) to be a valid pdf, it must integrate to one. The region under \(f_X\) consists of a rectangle of area \(2b(1)\) and two triangles of area \(c(1)/2\) each, so its total area is \(2b + c\); therefore, \(2b + c = 1\).

By inspection of the figure, \(F_X(a-b)\) is the area of the first triangle, so \(F_X(a-b) = \frac{c}{2}\). By the formula given for \(F_X\), \(F_X(a-b) = \frac{5c^2}{6}\). So \(\frac{c}{2} = \frac{5c^2}{6}\), or \(c = 0.6\). (ALTERNATIVELY, taking the derivative of the expression given form \(F_X\) shows that \(f_X(u) = \frac{5}{3}(u - (a - b - c))\) in the interval \([a-b-c, a-b]\). Setting \(f_X(a-b) = 1\) gives \(\frac{5c}{3} = 1\) or \(c = 0.6\). ALTERNATIVELY, taking the derivative of \(F_X\) twice yields \(F''_X = f'_X = \frac{5}{3}\) over the interval \([a-b-c, a-b]\). By the picture, the slope of \(f_X\) over the interval is \(\frac{1}{c}\). So \(\frac{5}{3} = \frac{1}{c}\) or \(c = 0.6\). There are other ways to derive \(c = 0.6\); they all involve comparing the picture of \(f_X\) over the interval \([a-b-c, a-b]\) to the formula given for \(F_X\).) Since \(2b + c = 1\), \(b = 0.2\).

4. (a)

\[
F_T(t) = 1 - e^{-\int_0^t h(s)ds} = \begin{cases} 
1 - e^{-t} & \text{if } 0 < t < 1 \\
1 - e^{1-t} & \text{if } t \geq 1
\end{cases}
\]

(b) No. For example, \(P(T > 1) = 1 - F_T(1) = e^{-1}\), whereas

\[
P(T > 2 | T > 1) = \frac{P(T > 2, T > 1)}{P(T > 1)} = \frac{P(T > 2)}{P(T > 1)} = \frac{1 - F_T(2)}{1 - F_T(1)} = \frac{e^{-3}}{e^{-1}} = e^{-2}
\]
5. (a) For $-1 \leq u \leq 1$ the likelihood ratio is given by $\Lambda(u) = 2|u|$. The MAP rule declares $H_1$ if $\Lambda(u) \geq \frac{\pi_0}{\pi_1} = \frac{1}{2}$, i.e., if $|u| \geq \frac{1}{4}$, and declares $H_0$ if $|u| < \frac{1}{4}$.

(b) 
\[ p_{\text{false alarm}} = \int_{\frac{1}{4} \leq |u| \leq 1} f_0(u) \, du = \int_{\frac{1}{4} \leq |u| \leq 1} \frac{1}{2} \, du = 2 \int_{\frac{1}{4} \leq u \leq 1} \frac{1}{2} \, du = \frac{3}{4} \]

6. (a) No, because, for example, the support is not a product set.

(b) 
\[ f_X(u) = \begin{cases} f_u \frac{3}{2} \, dv = \frac{3}{2} (1 - u^2), & 0 \leq u \leq 1 \\ 0, & u < 0 \text{ or } u > 1 \end{cases} \]

(c) For $0 \leq u < 1$. (We will not worry in grading whether the values $u = 0$ or $u = 1$ are included or not.)

(d) For $0 \leq u < 1$,
\[ f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(u)} = \begin{cases} \frac{1}{1-u^2}, & u^2 \leq v \leq 1 \\ 0, & v < u^2 \text{ or } v > 1 \end{cases} \]

That is, for $0 \leq u < 1$, given $X = u$, $Y$ is uniformly distributed over the interval $[u^2, 1]$.

(e) 
\[ P\{Y > X\} = \int_0^1 \left[ \int_u^1 \frac{3}{2} \, dv \right] \, du = \frac{3}{4} \]

or simply \( \frac{3}{2} \times \text{Area} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4} \)