University of Illinois Fall 2011

ECE 313: Final Exam

1. **[8 points]** Suppose that you have three biased coins. Coins C_1 and C_2 have $P\{heads\} = 1/4$, and coin C_3 has $P\{heads\} = 1/3$. You grab one coin at random (all being equally likely to be chosen), flip it, record the outcome (whether it was heads or tails), and put the coin aside. Then you grab one of the remaining two coins at random (all being equally likely to be chosen), flip it, and record the outcome (whether it was heads or tails).

   (a) **[2 points]** Find $P(C_3C_1)$, where C_3C_1 is the event that coin C_3 is grabbed first and coin C_1 is grabbed second.

   (b) **[2 points]** Find $P(T_1H_2|C_3C_1)$, where T_1H_2 is the event where the first flip shows tails and the second flip shows heads.

   (c) **[4 points]** Let X be a random variable that indicates the total number of heads that show in the two flips. Find the pmf of X.

2. **[8 points]** The two parts of this problem are unrelated to each other.

   (a) **[4 points]** Consider the following three functions:

$$\begin{aligned} f(x) &= \ln(x) && x \in (1/2, 3/4), \\ g(x) &= 4 && x \in (1/2, 3/4), \\ h(x) &= 2e^{-2x} && x \geq 0. \end{aligned}$$

   Which one(s) are valid pdfs? Mark True or False below, providing brief explanations for your thinking.

   (b) **[4 points]** Consider the following three functions:

$$\begin{aligned} f(x,y) &= \frac{3}{\sqrt{2\pi}} \exp\left(-3x - \frac{y^2}{2}\right) && x \geq 0, y \in (-\infty, \infty), \\ g(x,y) &= 2\sin(x) && x \in (-\pi, 0), y \in (0, 1/2), \\ h(x,y) &= 2 && x \in (1/4, 3/4), y \in (0, 1). \end{aligned}$$

   Which one(s) are valid joint pdfs? Mark True or False below, providing brief explanations for your thinking.

3. **[8 points]** Let X and Y be two continuous-type random variables with joint pdf $f_{X,Y}(u,v) = \frac{1}{2}e^{-u}$ for $u \geq 0$ and $0 \leq v \leq 2$, and 0 otherwise.

   (a) **[2 points]** Find the marginal pdf of X.

   (b) **[2 points]** Find the conditional pdf $f_{X|Y}(u|v)$ for all u, v.

   (c) **[4 points]** Let $Z = X + Y$. Find the pdf of Z.

4. **[8 points]** Suppose that you have three biased coins and a fair die. Coin C_1 has $P\{heads\} = 1/3$, while coins C_2 and C_3 have $P\{heads\} = 1/4$. Each part of this problem is unrelated to the other parts of the problem.

   (a) **[4 points]** Flip coin C_2 4 times and then flip coin C_3 3 times. Let Y be the total number of heads that show. Find the pmf of Y.

   (b) **[4 points]** Roll the die and let N be the number showing. Then flip coin C_1 N times. Let Z be the number of heads showing. Find $\mathsf{E}[Z]$.

5. **[6 points]** The voltage received at a radio is modeled as a random variable $\mathbb{X}$. When there is no transmit signal, the voltage is uniformly distributed between -2 and 2 Volts. When there is a transmit signal, the voltage is uniformly distributed between 0 and +3 Volts. Suppose that a signal is transmitted with probability 0.75 (and no signal is transmitted with probability 0.25)

   (a) **[3 points]** Find the pdf of $\mathbb{X}$.

   (b) **[3 points]** Given that the voltage received $\mathbb{X} = 1.5$ Volts, calculate the probability that a signal was transmitted.

6. **[8 points]** Simon's backyard pond contains g goldfish and c catfish. All fish are equally likely to be caught.

   (a) **[4 points]** Suppose that Simon catches a total of k fish (no fish are thrown back). What is the probability of catching x goldfish?

   (b) **[4 points]** Now suppose that all k fish are returned to the pond, and he starts fishing again, and catches a total of m fish. Let A be the event that among the caught set of m fish, exactly 2 goldfish are included that were also caught in the first catch? (Assume that fish do not learn from experience.) Find $P(A)$.

7. **[8 points]** Professor May B. Right often has her science facts wrong, and answers each of her students questions incorrectly with probability $1/4$, independently of other questions. In each lecture May is asked either 1 or 2 questions with equal probability.

   (a) **[2 points]** Let A be the event that May gives wrong answers to all the questions she gets in a given lecture. Find $P(A)$.

   (b) **[2 points]** Let B be the event that May gave wrong answers to all the questions she got in a given lecture, and let C be the event that she got two questions. Find $P(C|B)$.

   (c) **[4 points]** Let $\mathbb{X}$ and $\mathbb{Y}$ be the number of questions May gets and the number of questions she answers correctly in a lecture, respectively. Find the pmf of $\mathbb{Y}$.

8. **[8 points]** A car dealer sells two models of cars, Ray and Sprint. A Ray car breaks after time $\mathbb{X}$ days, where $\mathbb{X}$ is exponentially distributed with parameter λ_x. Similarly, a Sprint car breaks after a random time $\mathbb{Y}$ that is exponentially distributed with parameter λ_y. The random variables $\mathbb{X}$ and $\mathbb{Y}$ are independent.

   (a) **[4 points]** Given that a Ray did not break in the first u days, what is the expected time before the Ray breaks?

   (b) **[4 points]** What is the probability that a Ray breaks before a Sprint?

9. **[8 points]** A pipe-smoking mathematician carries, at all times, 2 matchboxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match he takes out the matchbox from his left-hand pocket with probability p, and from his right-hand pocket with probability $1-p$. Being a mathematician, he is oblivious to the remaining number of matches in the box. Consider the moment when the mathematician needs a match and first discovers that his left-hand matchbox is empty. Assume that both matchboxes initially contained N matches.

   (a) **[4 points]** The probability that there are exactly k matches in the right-hand box, $k = 0, 1, \cdots, N$, can be expressed as follows:

   $$P_k = \binom{M}{N} p^{N+1}(1-p)^{N-k}.$$

   Find M in terms of N and k.

(b) [**4 points**] The mathematician opens the right-hand matchbox and finds j matches in it. What is the maximum likelihood estimate of p, as a function of N and j?

10. [**8 points**] Consider a Poisson process of rate λ.

    (a) [**2 points**] What is the probability that there are no arrivals in the interval $[0, 2]$?

    (b) [**3 points**] What is the probability that there are two or fewer arrivals in the interval $[1, 3]$?

    (c) [**3 points**] Given that there are three arrivals during $[0, 3]$, what is the probability that there are two arrivals during $[0, 1]$?

11. [**8 points**] Suppose X and Y are mutually independent Gaussian random variables with $\mu_X = 2$, $\mu_Y = 0$, $\mathrm{Var}(X) = 4$ and $\mathrm{Var}(Y) = 5$. Let $Z = X + Y$.

    (a) [**3 points**] Find the numerical value of $P(Z < 0.5)$.

    (b) [**5 points**] Find the linear estimator $L^*(Z)$ of X based on Z with the minimum MSE.

12. [**14 points**] (2 points per answer)
    In order to discourage guessing, 2 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score. There is no need to justify your answer.

    (a) A, B, C are three events such that $0 < P(A) < 1$, $0 < P(B) < 1$ and $0 < P(C) < 1$.

TRUE	FALSE	
☐	☐	$P(A\|B) + P(A^c\|B) = 1$.
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	$P(A\|B) = P(B\|A)$, then $P(A) = P(B)$.

    (b) Consider a binary hypothesis testing problem where the prior probability of hypothesis H_0 is π_0 and the prior probability of hypothesis H_1 is π_1. Denote the probabilities of false alarm and missed detection for the ML decision rule by P_{FA}^{ML} and P_{MD}^{ML}, respectively. Similarly, denote the probabilities of false alarm and missed detection for the MAP decision rule by P_{FA}^{MAP} and P_{MD}^{MAP}, respectively.

TRUE	FALSE	
☐	☐	$P_{FA}^{ML} + P_{MD}^{ML} = 1$.
☐	☐	$P_{FA}^{MAP} \leq P_{FA}^{ML}$.
☐	☐	$P_{FA}^{ML} \cdot \pi_0 + P_{MD}^{ML} \cdot \pi_1 \geq P_{FA}^{MAP} \cdot \pi_0 + P_{MD}^{MAP} \cdot \pi_1$.
☐	☐	If $\pi_0 = 0.5$ then $P_{MD}^{ML} = P_{MD}^{MAP}$.

6.4 Normal tables

Tables 6.1 and 6.2 below were computed using Abramowitz and Stegun, *Handbook of Mathematical Functions*, Formula 7.1.26, which has maximum error at most 1.5×10^{-7}.

Table 6.1: Φ function, the area under the standard normal pdf to the left of x.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Table 6.2: Q function, the area under the standard normal pdf to the right of x.

x	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010

x	**0.0**	**0.2**	**0.4**	**0.6**	**0.8**
0.0	0.5000000	0.4207403	0.3445783	0.2742531	0.2118553
1.0	0.1586553	0.1150697	0.0807567	0.0547993	0.0359303
2.0	0.0227501	0.0139034	0.0081975	0.0046612	0.0025552
3.0	0.0013500	0.0006872	0.0003370	0.0001591	0.0000724
4.0	0.0000317	0.0000134	0.0000054	0.0000021	0.0000008