University of Illinois Fall 2011

ECE 313: Midterm Exam I

Monday October 10, 2011
6:40 p.m. — 8:00 p.m.
1320 DCL (10 and 11 am sections) and 151 Everitt (noon and 1pm sections)

1. (a) There are 36 possible outcomes if you roll two dice simultaneously. Of these outcomes, 6 have both dice showing the same number: they both show $i \in \{1,2,3,4,5,6\}$. Of the remaining outcomes, 10 have number 2 showing on exactly one of the dice: pairs of the form $(2,i)$ or $(i,2)$ for $i \in \{1,3,4,5,6\}$. Therefore,
$P\{\text{two dice show different numbers and that neither die shows 2}\} = \frac{36-6-10}{36}\ \boxed{= \frac{5}{9}.}$

   (b) If all dice show different numbers then one can use total probability by conditioning on the number shown by one of the dice (say die A) and requiring the other two dice (say dice B and C) to be different among themselves and different from the chosen number. That is,
$P\{\text{all dice show different numbers}\}$
$= \sum_{i=1}^{6} P\{\text{die } B \neq \text{die } C \text{ and neither shows } i|\text{die } A = i\}P\{\text{die } A = i\} = \sum_{i=1}^{6} \left(\frac{5}{9}\right)\left(\frac{1}{6}\right)$
$\boxed{= \frac{5}{9}.}$

2. (a) Using the law of total probability

$$\begin{aligned}
& P\{\text{Sara consumes the treat}\} \\
= \ & P\{\text{Sara consumes the treat}|\text{treat is M\& M bar}\} \cdot P\{\text{treat is M\& M bar}\} \\
& \quad + P\{\text{Sara consumes the treat}|\text{treat is raisin box}\} \cdot P\{\text{treat is raisin box}\} \\
= \ & \frac{1}{5}\cdot\frac{3}{4} + \frac{2}{5}\cdot\frac{1}{4} \\
= \ & \frac{1}{4}.
\end{aligned}$$

   (b) Using Bayes rule

$$\begin{aligned}
& P[\text{treat is raisin box}|\text{Sara consumes the treat}] = \\
& \qquad \frac{P\{\text{Sara consumes the treat}|\text{treat is raisin box}\} \cdot P\{\text{treat is raisin box}\}}{P\{\text{Sara consumes the treat}\}} \\
= \ & \frac{\frac{2}{5}\cdot\frac{1}{4}}{\frac{1}{4}} \\
= \ & \frac{2}{5}.
\end{aligned}$$

3. (a) N is a Geometric random variable with parameter $\frac{1}{6}$. Thus $E[N] = 6$.

   (b) The event $N \geq 3$ is equivalent to stating that the first two throws didn't yield the number 6. Since each throw is independent of the other,

$$\begin{aligned}
P\{N = 2+k|N > 2\} &= \frac{P\{N = 2+k\}}{P\{N > 2\}} \\
&= P\{N = k\} \quad \text{by the memoryless property of Geometric r.v.} \\
&= \left(\frac{5}{6}\right)^{k-1}\frac{1}{6}.
\end{aligned}$$

Hence the expected value of N given that $N > 2$ is, by definition,

$$\begin{aligned} E[N|N>2] &= \sum_{k=1}^{\infty}(k+2)\cdot P\{N=2+k|N>2\} \\ &= \sum_{k=1}^{\infty}(k+2)\cdot P\{N=k\} \\ &= 2+E[N] \\ &= 8. \end{aligned}$$

4. (a)

$$P\{M=k\} = \binom{k}{1}\left(\frac{5}{6}\right)^{k-1}\frac{1}{6}\cdot\frac{1}{6}.$$

Substituting $k=3$ we get the answer of $\frac{5^2}{2\cdot 6^3} = \frac{25}{432}$.

(b) Using LOTUS

$$\begin{aligned} E\left[\frac{1}{M}\right] &= \sum_{k=1}^{\infty}\frac{1}{k}\cdot P\{M=k\} \\ &= \sum_{k=1}^{\infty}\frac{1}{36}\left(\frac{5}{6}\right)^{k-1} \\ &= \frac{1}{6}. \end{aligned}$$

5. (a) It is easy to see that $\mathbb{X}$ can take on values 0, 40, 200, and 240.

(b) The link between Springfield and Champaign-Urbana fails when the direct link fails *and* at least one of the two upper links (Springfield-Decatur and Decatur-Champaign-Urbana) fails. The probability of the former event is $\frac{1}{2}$ and the probability of the latter event is $\frac{3}{4}$. So the probability of outage is the product of these two probabilities: $\frac{3}{8}$.