ECE 313: Problem Set 13
Minimum MSE estimation, joint Gaussian random variables

1. **[Linear minimum MSE estimation from uncorrelated observations]**
   Suppose Y is estimated by a linear estimator, $L(X_1, X_2) = a + bX_1 + cX_2$, such that X_1 and X_2 have mean zero and are uncorrelated with each other.

   (a) Determine a, b and c to minimize the MSE, $E[(Y - (a + bX_1 + cX_2))^2]$. Express your answer in terms of $E[Y]$, the variances of X_1 and X_2, and the covariances $\mathrm{Cov}(Y, X_1)$ and $\mathrm{Cov}(Y, X_2)$.

   (b) Express the MSE for the estimator found in part (a) in terms of the variances of X_1, X_2, and Y and the covariances $\mathrm{Cov}(Y, X_1)$ and $\mathrm{Cov}(Y, X_2)$.

2. **[What's new? or the innovations method]**
   The previous problem shows that if Y is to be estimated by a linear combination of X_1 and X_2, the solution is particularly simple if $\mathrm{Cov}(X_1, X_2) = 0$. Intuitively, that is because, even if X_1 is already known, X_2 offers completely new information. If $\mathrm{Cov}(X_1, X_2) \neq 0$, then by preprocessing, we can reduce the problem to the previous case. The idea is to let $\widetilde{X_2} = X_2 - hX_1$, where the constant h is chosen so that X_1 and $\widetilde{X_2}$ are uncorrelated. Intuitively, $\widetilde{X_2}$ is the part of X_2 that is new, or innovative, to someone who already knows X_1. Then, the best linear estimator of Y given X_1 and $\widetilde{X_2}$ can be easily found by the method of the previous problem. But a linear estimator based on X_1 and $\widetilde{X_2}$ can also be expressed as a linear estimator based on X_1 and X_2, or vice versa. So the estimator would also be the best linear estimator of Y given X_1 and X_2.

   This recipe is illustrated by an example as follows. Suppose Y, X_1, and X_2 have mean zero and variance one, and suppose $\mathrm{Cov}(Y, X_1) = \mathrm{Cov}(Y, X_2) = 0.8$ and $\mathrm{Cov}(X_1, X_2) = 0.5$.

   (a) Find h so that X_1 and $\widetilde{X_2} = X_2 - hX_1$ are uncorrelated.

   (b) Find $\mathrm{Var}(\widetilde{X}_2)$ and $\mathrm{Cov}(Y, \widetilde{X}_2)$.

   (c) Appealing to the previous problem, find the values of a, b, and c so that the MSE for estimating Y by the linear estimator $a + bX_1 + c\widetilde{X}_2$ is minimized, and find the resulting MSE.

   (d) Express the linear estimator found in part (c) as a linear combination of X_1 and X_2.

3. **[An estimation problem]**
Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{8uv}{(15)^4} & u \geq 0, v \geq 0, u^2+v^2 \leq (15)^2 \\ 0 & \text{else} \end{cases}$$

(a) Find the constant estimator, δ^*, of Y, with the smallest mean square error (MSE), and find the MSE.

(b) Find the unconstrained estimator, $g^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE.

(c) Find the linear estimator, $L^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE. (Hint: You may use the fact $E[XY] = \frac{75\pi}{4} \approx 58.904$, which can be derived using integration in polar coordinates.)

4. **[Jointly Gaussian Random Variables]**
Suppose X and Y are jointly Gaussian random variables with $E[X] = 2$, $E[Y] = 4$, $\mathrm{Var}(X) = 9$, $\mathrm{Var}(Y) = 25$, and $\rho = 0.2$. Let $W = X + 2Y + 3$.

(a) Find $E[W]$ and $\mathrm{Var}(W)$.

(b) Calculate the numerical value of $P\{W \geq 20\}$.

(c) Find the unconstrained estimator $g^*(W)$ of Y based on W with the minimum MSE, and find the resulting MSE.