University of Illinois Fall 2010

ECE 313: Final Exam

Friday, December 10, 2011, 8:00 a.m. — 11:00 a.m.
Room 151 Everitt Lab & Room 149 National Soybean Research Center

1. (a) $P(S) = 0.36, P(A|S) = 0.22, P(A) = 0.30.$ $P(AS) = (0.22)(0.36) = 0.0792.$
   (b) $P(S|A) = P(AS)/P(A) = 0.0792/0.3 = 0.264$

2.

$$\begin{aligned} P\{(X+Y)^2 < 9\} &= P\{-3 < X+Y < 3\} = P\left\{\frac{-3-1}{2} < \frac{X+Y-1}{2} < \frac{3-1}{2}\right\} \\ &= \Phi(1) - \Phi(-2) = \Phi(1) + \Phi(2) - 1. \end{aligned}$$

3. X and Y are jointly Gaussian so that Z and X are jointly Gaussian, so the minimum MSE estimator is linear.

$$\begin{aligned} g^*(Z) &= L^*(Z) = \mu_X + \frac{\mathrm{Cov}(X, X+Y)}{\mathrm{Var}(X+Y)}(Z - \mu_X - \mu_Y) \\ &= \mu_X + \frac{\mathrm{Var}(X)}{\mathrm{Var}(X) + \mathrm{Var}(Y)}(Z - \mu_X - \mu_Y) \\ &= 2 + \frac{0.5}{2.5}(Z-2) = 2 + \frac{Z-2}{5}. \end{aligned}$$

4. (a) The probability mass function $p_X(k)$ for $k = 0, 1, 2$ is given by:

$$\begin{aligned} p_X(0) &= P\{\text{second ball is white}\} = \left(\frac{5}{8}\right)\left(\frac{6}{10}\right) = \frac{3}{8} \\ p_X(1) &= P\{\text{second ball is red}\} = \left(\frac{5}{8}\right)\left(\frac{4}{10}\right) + \left(\frac{3}{8}\right)\left(\frac{6}{10}\right) = \frac{19}{40} \\ p_X(2) &= P\{\text{second ball is black}\} = \left(\frac{3}{8}\right)\left(\frac{4}{10}\right) = \frac{3}{20} \end{aligned}$$

Let H_0 be the hypothesis the first ball is white and H_1 be the hypothesis the first ball is black.

(b) If $X = 0$ then $\widehat{H}_{ML} = H_0$ because $P(X = 0|H_0) = 0.6 > 0 = P(X = 0|H_1)$.
If $X = 1$ then $\widehat{H}_{ML} = H_1$ because $P(X = 1|H_0) = 0.4 < 0.6 = P(X = 1|H_1)$.
If $X = 2$ then $\widehat{H}_{ML} = H_1$ because $P(X = 2|H_0) = 0 < 0.4 = P(X = 2|H_1)$. Equivalently, the likelihood ratio function is given by

$$\Lambda(k) = \begin{cases} 0 & k = 0 \\ 1.5 & k = 1 \\ +\infty & k = 2, \end{cases}$$

and the ML rule decides in favor of H_1 if $\Lambda(k) \geq 1$, or equivalently, if $k \in \{1, 2\}$. Thus, the ML rule is given by:

X	$\widehat{H}_{ML}$
0	H_0
1	H_1
2	H_1

(c) The mechanism used to determine which hypothesis is really true in the first place is the drawing from the grey bucket; H_0 has prior probability $\pi_0 = \frac{5}{8}$ and H_1 has prior probability $\pi_1 = \frac{3}{8}$. Since the MAP rule is the likelihood ratio test with threshold $\tau = \frac{\pi_0}{\pi_1} = \frac{5}{3}$, in view of the expression for Λ above, the MAP rule declares H_1 is true for $X = 2$, and declares H_0 otherwise. Thus, the MAP rule is given by

X	$\widehat{H}_{MAP}$
0	H_0
1	H_0
2	H_1

5. (a) Network fails if links 1 and 2 both fail, or if link 3 fails. Thus, $F = (F_1F_2) \cup F_3$. Hence,

$$\begin{aligned} P(F) &= P((F_1F_2) \cup F_3) \\ &= P(F_1F_2) + P(F_3) - P(F_1F_2F_3) \\ &= p_1p_2 + p_3 - p_1p_2p_3 \end{aligned}$$

(b) The pmf of network capacity C is given by:

$$\begin{aligned} p_C(0) &= P\{F\} = \frac{1}{4} + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) - \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{17}{32} \\ p_C(10) &= P(F_1^cF_2F_3^c) = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right) = \frac{9}{32} \\ p_C(20) &= P(F_1F_2^cF_3^c) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{32} \\ p_C(30) &= P(F_1^cF_2^cF_3^c) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{3}{32} \end{aligned}$$

Check: $\sum_k p_C(k) = 1$.

(c) The lifetime, T, of the network satisfies

$$\begin{aligned} 1 - F_T(t) = P\{T > t\} &= P(\{T_1 > t\} \cup \{T_2 > t\}) \\ &= P\{T_1 > t\} + P\{T_2 > t\} - P(\{T_1 > t\}\{T_2 > t\}) \\ &= e^{-\lambda t} + e^{-\lambda t} - e^{-2\lambda t} \\ &= 2e^{-\lambda t} - e^{-2\lambda t} \end{aligned}$$

Taking the derivative of both sides; $f_T(t) = 2\lambda(e^{-\lambda t} - e^{-2\lambda t})$. Hence,

$$h_T(t) = \frac{2\lambda(e^{-\lambda t} - e^{-2\lambda t})}{2e^{-\lambda t} - e^{-2\lambda t}} = \frac{2\lambda(1 - e^{-2\lambda t})}{2 - e^{-\lambda t}}$$

(d) Let T represent the lifetime of the network consisting of nodes B and C. Then,

$$\begin{aligned} 1 - F_T(t) = P\{T > t\} &= P(\{T_1 > t\}\{T_2 > t\}) \\ &= P\{T_1 > t\}P\{T_2 > t\} = e^{-\lambda t}e^{-\lambda t} = e^{-2\lambda t} \end{aligned}$$

Thus, $f_T(t) = 2\lambda e^{-2\lambda t}$ (i.e. T is exponentially distributed with parameter 2λ) and

$$h_T(t) = \frac{2\lambda e^{-2\lambda t}}{e^{-2\lambda t}} = 2\lambda.$$

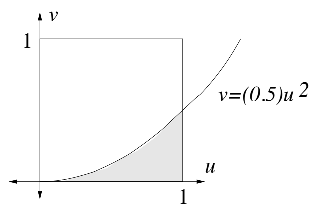
6. (a) $\frac{\binom{5}{2}}{\binom{7}{4}} = \frac{\binom{5}{2}}{\binom{7}{3}} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 7 \cdot 6 \cdot 5} = \frac{2}{7}$

(b) $\frac{2 \cdot 2 \cdot 3}{\binom{7}{4}} = \frac{12}{\binom{7}{3}} = \frac{12 \cdot 3 \cdot 2}{7 \cdot 6 \cdot 5} = \frac{12}{35}$

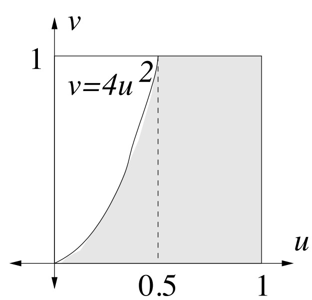
(c) $\frac{1}{\binom{5}{2}} = \frac{1}{10}$

7. (a) $E[X + 2Y + 3Z + 4] = E[X] + 2E[Y] + 3E[Z] + 4 = 28.$
   (b) $\mathrm{Var}(X + 2Y + 3Z + 4) = \mathrm{Var}(X) + 2^2\mathrm{Var}(Y) + 3^2\mathrm{Var}(Z) = (14)(9) = 126.$
   (c) $\mathrm{Cov}(X + 2Y, 3Z + 4) = 0.$
   (d) $E[X + YZ] = E[X] + E[Y]E[Z] = 20.$
   (e) $E[X^2] = E[X]^2 + \mathrm{Var}(X) = 25$; $E[(XY)^2] = E[X^2]E[Y^2] = 625.$
   (f) $E[XY] = E[X]E[Y] = 16$; $\mathrm{Var}(XY) = E[(XY)^2] - E[XY]^2 = 625 - 256 = 369.$

8. (a) $P\{Z \le 0.5\} = P\{Y \le (0.5)X^2\} = \int_0^1 (0.5)u^2 du = \frac{1}{6}.$



   (b) $P\{Z \le 4\} = P\{Y \le 4X^2\} = \int_0^{0.5} 4u^2 du + 0.5 = \frac{2}{3}.$



9. (a) False, False, True, True
   (b) True, True, False, True
   (c) True, True