University of Illinois Fall 2010

ECE 313: Final Exam

Friday, December 10, 2011, 8:00 a.m. — 11:00 a.m.
Room 151 Everitt Lab & Room 149 National Soybean Research Center

1. [**20 points**] A town has two supermarkets, Spar and Aldi; 36 percent of people shop at Spar; 22 percent of people who shop at Spar also shop at Aldi; 30 percent of people shop at Aldi.

   (a) [**10 points**] What is the probability a randomly selected person shops at both Aldi and Spar?

   (b) [**10 points**] What is the conditional probability a randomly selected person shops at Spar given that she/he shops at Aldi?

2. [**10 points**] Suppose $Z = (X+Y)^2$, where X and Y are independent, Gaussian random variables with $\mu_X = 0$, $\mu_Y = 1$, $\text{Var}(X) = 1$ and $\text{Var}(Y) = 3$. Find $P(Z < 9)$, expressing your answer in terms of the Φ function.

3. [**20 points**] Suppose X and Y are mutually independent Gaussian random variables with $\mu_X = 2$, $\text{Var}(X) = 0.5$, $\mu_Y = 0$, and $\text{Var}(Y) = 2$. Let $Z = X + Y$. Find the unconstrained estimator $g^*(Z)$ of X based on Z with the minimum MSE.

4. [**30 points**] There are three buckets: one grey, one white, and one black.
   The grey bucket contains 5 white and 3 black balls.
   The white bucket contains 6 white and 4 red balls.
   The black bucket contains 4 black and 6 red balls.
   A ball is first drawn from the grey bucket. The color of that ball determines which bucket a second ball is drawn from; the second ball is drawn from the bucket with the same color as the first ball. For example, if the first ball is white, then the second ball is drawn from the white bucket.
   Let $X = 0$ if the second ball is white, $X = 1$ if the second ball is red, $X = 2$ if the second ball is black.

   (a) [**12 points**] Derive the probability mass function (pmf) of X.

   Let H_0 be the hypothesis the first ball is white and H_1 be the hypothesis the first ball is black.

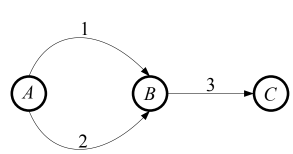
   (b) [**9 points**] Derive the maximum likelihood rule for selecting a hypothesis given X. Express your answer by filling in each blank cell in the table below with either H_0 or H_1.

X	$\widehat{H}_{ML}$
0	
1	
2	

   (c) [**9 points**] Derive the MAP rule for selecting a hypothesis given X. Express your answer by filling in each blank cell in the table below with either H_0 or H_1.

X	$\widehat{H}_{MAP}$
0	
1	
2	

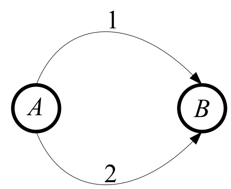
5. [**30 points**] Consider the network shown below, with source node A and terminal node C. Each link is assumed to fail independently of the others. The link failure probabilities are: p_1, p_2, and p_3. The link capacities are: C_1, C_2 and C_3.



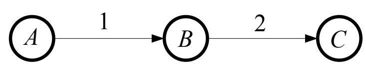
   (a) [**6 points**] Express $P(F)$ in terms of p_1, p_2 and p_3, where F is the event that the network fails.

(b) **[8 points]** Determine the pmf $p_C(k)$ of network capacity C if $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{4}$, and $p_3 = \frac{1}{4}$, and $C_1 = 10$, $C_2 = 20$, and $C_3 = 30$.

(c) **[8 points]** Determine the failure rate of the network shown below, if the failure rate function of each link is λ for all $t \geq 0$. (The network fails when both links have failed.)



(d) **[8 points]** Determine the failure rate of the network shown below, if the failure rate function of each link is λ for all $t \geq 0$. (The network fails when at least one of the likes fails.)



6. **[25 points]** Four students are selected from a class of seven students to get free concert tickets, with all sets of size four being equally likely. Two of the students are brothers of each other, and two of the students are sisters of each other. Find the probabilities indicated. *Full credit will be given only if you show your work and express the correct numerical answers as fractions in reduced form.*

   (a) **[9 points]** $P\{\text{both brothers selected}\}$.

   (b) **[8 points]**$P\{\text{exactly one of the brothers is selected and exactly one of the sisters is selected}\}$.

   (c) **[8 points]**$P(\text{both brothers selected}|\text{both sisters selected})$.

7. **[30 points]** Suppose X, Y, and Z are mutually independent random variables, such that each has mean four and variance nine. Find the numerical values of the quantities indicated. Check your answers carefully.

   (a) **[5 points]** $E[X + 2Y + 3Z + 4]$.

   (b) **[5 points]** $\text{Var}(X + 2Y + 3Z + 4)$.

   (c) **[5 points]** $\text{Cov}(X + 2Y, 3Z + 4)$.

   (d) **[5 points]** $E[X + YZ]$.

   (e) **[5 points]** $E[X^2Y^2]$.

   (f) **[5 points]** $\text{Var}(XY)$.

8. **[30 points]** Let $Z = \frac{Y}{X^2}$, such that X and Y have joint pdf given by

$$f_{X,Y}(u,v) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1, 0 \leq v \leq 1 \\ 0 & \text{else.} \end{cases}$$

   (a) **[15 points]** Find the numerical value of $P\{Z \leq 0.5\}$. Also, sketch a region in the plane so that the area of the region is $P\{Z \leq 0.5\}$.

   (b) **[15 points]** Find the numerical value of $P\{Z \leq 4\}$. Also, sketch a region in the plane so that the area of the region is $P\{Z \leq 4\}$.

9. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE	FALSE	
☐	☐	$P(A\|B) + P(B\|A) = 1.$
☐	☐	$P(AB) + P(A \cup B) = 2(P(A) + P(B)).$
☐	☐	$P(AB) \leq P(A).$
☐	☐	$P(AB) \leq P(A\|B).$

(b) Suppose the random variables X and Y are nonnegative, continuous-type random variables that are independent.

TRUE	FALSE	
☐	☐	$P\{\|X - Y\| \geq 2\} \leq E[(X - Y)^2]$
☐	☐	$g(X)$ is uniformly distributed over the interval $[0, 1]$, where $g(u) = F_X(u)$.
☐	☐	$E\left[\frac{X}{Y}\right] = \frac{E[X]}{E[Y]}.$
☐	☐	$P\{X \geq Y\} + P\{\frac{1}{X} \geq \frac{1}{Y}\} = 1.$

(c) Suppose $(N_t : t \geq 0)$ is a Poisson random process with rate $\lambda = 2$.

TRUE	FALSE	
☐	☐	$P(N_5 - N_2 = 3\|N_2 = 5) = 36e^{-6}$
☐	☐	$P(N_2 = 5\|N_5 - N_2 = 3) = \frac{4^5 e^{-4}}{5!}$