University of Illinois Fall 2010

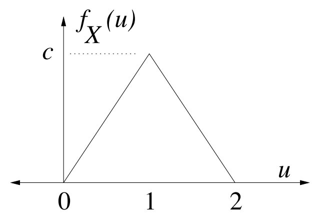
ECE 313: Hour Exam II

Monday November 15, 2010
7:00 p.m. — 8:00 p.m.
124 Burrill Hall & Room 100 Noyes Lab

1. (a) $P\{T \geq t\} = F_T^c(t) = e^{-\lambda t} = e^{-(\ln 2)t} = 2^{-t}$.

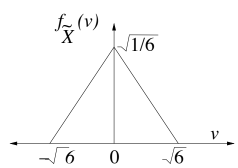
   (b) $P(T \leq 1 | T \leq 2) = \frac{P\{T \leq 1, T \leq 2\}}{P\{T \leq 2\}} = \frac{P\{T \leq 1\}}{P\{T \leq 2\}} = \frac{1-\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

   Suppose X has the pdf shown:



   (c) The area under the pdf is one half the base of the triangle times the height, equal to c. So $c = 1$.

   (d) To get mean zero, let $a = E[X] = 1$. Subtracting a causes the pdf to be centered at zero. And $b = \sigma_X = \sqrt{\frac{1}{6}}$. So $\widetilde{X} = \sqrt{6}(X-1)$. The support of $\widetilde{X}$ is thus the interval $(-\sqrt{6}, \sqrt{6})$, and the shape of the pdf is the same triangular shape that f_X has. So the pdf of $\widetilde{X}$ is the following:



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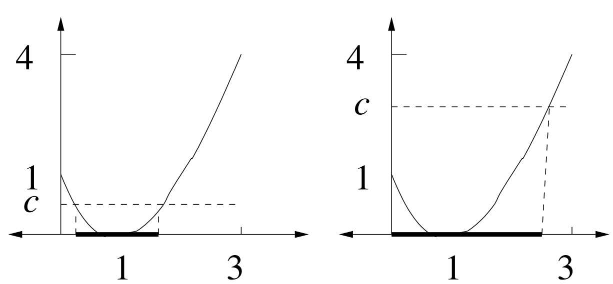
2. (a) Since X ranges over the interval $[0, 3]$, Y ranges over the interval $[0, 4]$.

   (b) The expression for $F_Y(c)$ is qualitatively different for $0 \leq c \leq 1$ and $1 \leq c \leq 4$, as seen in the following sketch:



   In each case, $F_Y(c)$ is equal to one third the length of the shaded interval. For $0 \leq c \leq 1$,

$$F_Y(c) = P\{(X-1)^2 \leq c\} = P\{1 - \sqrt{c} \leq X \leq 1 + \sqrt{c}\} = \frac{2\sqrt{c}}{3}.$$

For $1 \le c \le 4$,

$$F_Y(c) = P\{(X-1)^2 \le c\} = P\{0 \le X \le 1+\sqrt{c}\} = \frac{1+\sqrt{c}}{3}.$$

Combining these observations yields:

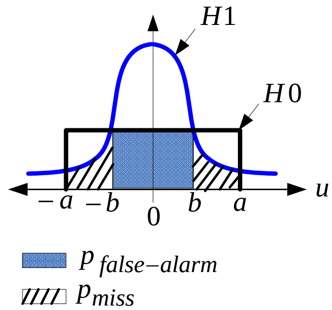
$$F_Y(c) = \begin{cases} 0 & c < 0 \\ \frac{2\sqrt{c}}{3} & 0 \le c < 1 \\ \frac{1+\sqrt{c}}{3} & 1 \le c < 4 \\ 1 & c \ge 4 \end{cases}$$

(c) By LOTUS,

$$E[Y] = E[(X-1)^2] = \int_0^3 (u-1)^2 \frac{1}{3} du = 1$$

3. (a) From the figure, the pdf for H_1 is smaller than the pdf for H_0 precisely when $b < |u| < a$. Thus, the ML rule is given by:

$$\widehat{H} = \begin{cases} H_0 & b < |X| < a \\ H_1 & \text{otherwise} \end{cases}$$



(b)

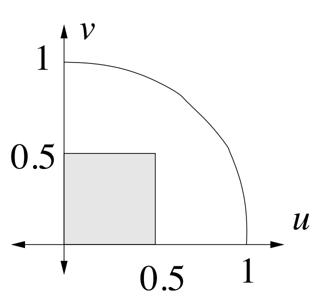
(c) These probabilities are calculated as follows:

$$\begin{aligned} p_{false\ alarm} &= \frac{2b}{2a} = \frac{b}{a} \\ p_{miss} &= 2(\Phi(a) - \Phi(b)) = 2(Q(b) - Q(a)) \end{aligned}$$

4. (a) No. For example, the support is not a product set because $(0.2, 0.8)$ and $(0.8, 0.2)$ are in the support of $f_{X,Y}$ but $(0.8, 0.8)$ is not.

(b) $P\{X \le Y\} = P\{Y \le X\}$ by symmetry and $P\{X \le Y\} + P\{Y \le X\} = 1$ (because $P\{X = Y\} = 0$) so $P\{X \le Y\} = 0.5$.

(c) The set $\{X \le 0.5, Y \le 0.5\}$ intersected with the support of $f_{X,Y}$ is the square region shown:



Thus, $P\{X \leq 0.5, Y \leq 0.5\} = \int_0^{0.5} \int_0^{0.5} 8uv \; du \; dv = \frac{1}{8}$.

(d) By LOTUS, $E[\frac{1}{XY}] = \int \int \frac{1}{uv} f_{X,Y}(u,v) du dv = \int \int_{\text{support}} 8 \; du \; dv = 2\pi$.