University of Illinois **Fall 2009**

ECE 313: Problem Set 14
Conditional Distributions; Covariance and Correlation

Due:	Wednesday December 9 at 4 p.m.
Reading:	Ross, Chapter 7, Sections 1-5; Chapter 8, Sections 1-4 Powerpoint Lecture Slides, Sets 35-38
Noncredit exercises:	Ross Chapter 7: Problems 1, 16, 26, 30 33, 34, 38; Theoretical Exercises: 1, 2, 17, 22, 23, 40 Chapter 8: problems 1-9, 15.
Reminder:	Due date for homework is Wednesdays at 4 p.m. again

1. **[Is gambling any good?]**
John is vacationing in Monte Carlo. Each evening the amount of money he takes to the casino is a random variable $\mathbb{X}$ with pdf
$$f_{\mathbb{X}}(u) = \begin{cases} Cu & 0 < u \leq \$100 \\ 0 & \text{else.} \end{cases}$$
At the end of each night, the amount $\mathbb{Y}$ he returns with is uniformly distributed between 0 and twice the amount he came to the casino with.
   (a) Find the value of the constant C.
   (b) For fixed α, $0 \leq \alpha \leq \$100$, what is $f_{\mathbb{Y}|\mathbb{X}}(v \mid \alpha)$, the conditional pdf of $\mathbb{Y}$ given that $\mathbb{X} = \alpha$?
   (c) If John goes to the casino with α dollars, what is the probability that he returns home with more than α dollars?
   (d) If John is a creature of habit and always goes to the casino with α dollars, what is his average profit from gambling? That is, what is $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = \alpha]$?
   (e) Determine the joint pdf $f_{\mathbb{X},\mathbb{Y}}(u, v)$.
   (f) Determine the marginal density $f_{\mathbb{Y}}(v)$.
   (g) Determine $\mathsf{E}[\mathbb{Y} - \mathbb{X}]$, the expected value of $\mathbb{Y} - \mathbb{X}$. This is the average profit/loss that John makes at the Casino.

2. **[Drill in computing correlations etc.]**
[Adapted from Ross, Problem 7.4] The random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf
$$f_{\mathbb{X},\mathbb{Y}}(u, v) = \begin{cases} \frac{1}{v}, & 0 < u < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$
   (a) Compute the means and variances of $\mathbb{X}$ and $\mathbb{Y}$.
   (b) Compute $\mathsf{cov}(\mathbb{X}, \mathbb{Y})$ and the correlation coefficient $\rho_{\mathbb{X},\mathbb{Y}}$.

3. **[Drill Problem on Correlation]**
Let $\mathsf{E}[\mathbb{X}] = 4, \mathsf{E}[\mathbb{Y}] = 1, \mathsf{var}(\mathbb{X}) = 1, \mathsf{var}(\mathbb{Y}) = 4$, and $\rho_{\mathbb{X},\mathbb{Y}} = 0.1$.
   (a) If $\mathbb{Z} = 2(\mathbb{X} + \mathbb{Y})(\mathbb{X} - \mathbb{Y})$, what is $\mathsf{E}[\mathbb{Z}]$?
   (b) If $\mathbb{T} = 2\mathbb{X} + \mathbb{Y}$ and $\mathbb{U} = 2\mathbb{X} - \mathbb{Y}$, what is $\mathsf{cov}(\mathbb{T}, \mathbb{U})$?
   (c) Find the mean and variance of $\mathbb{W} = 3\mathbb{X} + \mathbb{Y} + 2$.

4. **[Another drill problem]**
This problem has three independent parts. Do not apply the numbers from one part to the others.

(a) If $\mathsf{var}(\mathbb{X}+\mathbb{Y}) = 64$ and $\mathsf{var}(\mathbb{X}-\mathbb{Y}) = 36$, what is $\mathsf{cov}(\mathbb{X},\mathbb{Y})$? If you are also told that $\mathsf{var}(\mathbb{X}) = 3 \cdot \mathsf{var}(\mathbb{Y})$, what is $\rho_{\mathbb{X},\mathbb{Y}}$?

(b) If $\mathsf{var}(\mathbb{X}+\mathbb{Y}) = \mathsf{var}(\mathbb{X}-\mathbb{Y})$, are $\mathbb{X}$ and $\mathbb{Y}$ uncorrelated ?

(c) If $\mathsf{var}(\mathbb{X}) = \mathsf{var}(Y)$, are $\mathbb{X}$ and $\mathbb{Y}$ uncorrelated ?

5. **[Correlations again]**
Let $\mathbb{Z} = \mathbb{X} - \mathbb{Y}$ where $\mathbb{X}$ and $\mathbb{Y}$ are nonnegative random variables such that $\mathbb{X}\mathbb{Y} = 0$.

(a) Show that $\mathsf{cov}(\mathbb{X},\mathbb{Y}) \leq 0$.

(b) Show that $\mathsf{var}(\mathbb{Z}) \geq \mathsf{var}(\mathbb{X}) + \mathsf{var}(\mathbb{Y})$.