University of Illinois **Fall 2009**

ECE 313: Problem Set 13
Functions of Random Variables

Due:	Friday December 4 at 4 p.m.
Reading:	Ross, Chapter 6 except Sections 6.6 and 6.8;
	Powerpoint Lecture Slides, Sets 34-37
Noncredit Exercises:	**Chapter 6:** Problems 26, 28-30, 51, 54
	Theoretical Exercises 8, 14 ,22, 23, 33,; Self-Test Problems 3, 5, 6, 7, 12

1. [**A piece of cake? or a sheet cake with a piece missing?**]
   The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \dfrac{4}{3}, & 0 < u < 1,\ 0 < v < 1,\ \max\{u,v\} > \frac{1}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Sketch the u-v plane and indicate on it the region where $f_{\mathbb{X},\mathbb{Y}}(u,v)$ is nonzero.
   (b) Find the marginal pdf $f_{\mathbb{Y}}(v)$ of $\mathbb{Y}$.
   (c) What is $P\{\mathbb{Y} \leq \alpha\mathbb{X}\}$ where $0 < \alpha \leq 1$?
   (d) What is $P\{\mathbb{Y} \leq \alpha\mathbb{X}\}$ where $1 < \alpha < \infty$?
   (e) If $\mathbb{Z} = \mathbb{Y}/\mathbb{X}$, find $P\{\mathbb{Z} \leq \alpha\}$ for all $\alpha,\ 0 < \alpha < \infty$.
   (f) Find the pdf $f_{\mathbb{Z}}(\alpha)$. Be sure to specify the value of $f_{\mathbb{Z}}(\alpha)$ for all $\alpha,\ -\infty < \alpha < \infty$.

2. [**One function of two random variables**]
   The joint pdf of $\mathbb{X}$ and $\mathbb{Y}$ is given by $f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} 2u, & 0 < u < 1, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

   Find the pdf of $\mathbb{Z} = \mathbb{X}^2\mathbb{Y}$.

3. [**Independent exponential random variables**]
   Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be independent *exponential* random variables with parameters λ_1 and λ_2 respectively.

   (a) Find the pdf of $\mathbb{Y} = \mathbb{X}_1 + \mathbb{X}_2$ by convolving the pdfs of $\mathbb{X}_1$ and $\mathbb{X}_2$ for the case $\lambda_1 = \lambda_2 = \lambda$.
   (b) Find the pdf of $\mathbb{Z} = \min\{\mathbb{X}_1, \mathbb{X}_2\}$.

4. [**The Maxwell-Boltzmann density function**]
   This problem is a *lot* easier than it looks...

   (a) If $\mathbb{X}$ is $\mathcal{N}(0,\sigma^2)$, use the magic formula in Example 7b, Chapter 5.7 of Ross to show that $\mathbb{X}^2$ has a *gamma* pdf with parameter $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.
   (b) Now, suppose that $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ are *independent* $\mathcal{N}(0,\sigma^2)$ random variables. Then $\mathbb{X}^2, \mathbb{Y}^2$, and $\mathbb{Z}^2$ are independent gamma random variables with parameters $\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$. Use the comment immediately following the proof of Proposition 3.1 on p. 255 of Ross to *state* what is the type of pdf of $\mathbb{W} = \mathbb{X}^2 + \mathbb{Y}^2 + \mathbb{Z}^2$, and *write down* explicitly the exact function $f_{\mathbb{W}}(\alpha)$.
   (c) Prove that $\mathsf{E}[\mathbb{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!
   (d) In a physical application, $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathbb{H} = \frac{1}{2}m\mathbb{W}$ is the kinetic energy of the particle. It is an axiom of statistical mechanics that the *average* kinetic energy is $\mathsf{E}[\mathbb{H}] = \mathsf{E}[\frac{1}{2}m\mathbb{W}] = \frac{1}{2}m\mathsf{E}[\mathbb{W}] = \frac{3}{2}m\sigma^2 = \frac{3}{2}kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in $^\circ K$. (Note that the average energy is $\frac{1}{2}kT$ per dimension.) Show that the kinetic energy $\mathbb{H}$ has the Maxwell-Boltzmann pdf: $f_{\mathbb{H}}(\beta) = \dfrac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta}\exp\left(-\dfrac{\beta}{kT}\right)$ for $\beta \geq 0$.

(e) $\mathbb{V} = \sqrt{\mathbb{W}} = \sqrt{\mathbb{X}^2 + \mathbb{Y}^2 + \mathbb{Z}^2}$ is the *speed* of the molecule.

Show that the pdf of $\mathbb{V}$ is $f_{\mathbb{V}}(\gamma) = \dfrac{4}{\sqrt{\pi}}\left(\dfrac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2 \exp\left(-\dfrac{m\gamma^2}{2kT}\right)$, for $\gamma \geq 0$. cf. Theoretical Exercise 1 of Chapter 5.

(f) What is the average speed of the molecule?

5. **[Average of n independent random variables]**

You will have been taught that when making measurements with instruments in the laboratory, it is best to take several readings and average the values rather than making just a single measurement. This problem explores why this is a good idea, and also considers a case when this idea fails to work.

We model the measurements of a parameter μ as independent continuous random variables $\mathbb{X}_1$, $\mathbb{X}_2$, ..., $\mathbb{X}_n$. From Equation (3.2) in Section 6.3 of Ross, it follows that the pdf of their sum $\mathbb{Z} = \mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n$ is the *convolution of their pdfs,* and from Theorem 7.1 of Chapter 5 it follows that the pdf of the average $\mathbb{W} = \dfrac{\mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n}{n}$ is given by $f_{\mathbb{W}}(\beta) = n \cdot f_{\mathbb{Z}}(n\beta)$.

(a) As you learned in ECE 210, convolutions can be computed via Fourier transforms. Use the Fourier transform tables in your ECE 210 text (or other book in the library) to find the Fourier transform of a zero-mean Gaussian pdf $\frac{1}{\sigma\sqrt{2\pi}}\exp(-t^2/2\sigma^2)$.

(b) Use the time shift theorem of Fourier transform theory to find the Fourier transform of the pdf $\frac{1}{\sigma\sqrt{2\pi}}\exp(-(t-\mu)^2/2\sigma^2)$ from your answer to part (a).

(c) If each $\mathbb{X}_i$ is a Gaussian random variable with mean μ and variance σ^2, what is the Fourier transform of the pdf of $\mathbb{Z} = \mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n$?

(d) Use the result of part (c) to show that $\mathbb{Z}$ is a Gaussian random variable with mean $n\mu$ and variance $n\sigma^2$. Note that this is a special case of Proposition 3.2 in Chapter 6 of the textbook. In fact, the analysis that you have done is easily modified to prove Proposition 3.2 very straightforwardly in comparison to the proof in the textbook.

(e) Use the result of part (d) to deduce that $\mathbb{W} = \mathbb{Z}/n$ is a Gaussian random variable with mean μ and variance σ^2/n. Since most of the probability mass of a Gaussian random variable lies within ± 3 standard deviations of the mean, the value of $\mathbb{W}$, the average of n measurements, is very much more likely to be very close to μ than the value of any individual measurement $\mathbb{X}_i$.

Now suppose that the $\mathbb{X}_i$ are Cauchy random variables with identical pdfs of the form $\dfrac{1}{\pi(1+(t-\mu)^2)}$, $-\infty < t < \infty$ which have a peak at μ. We now essentially repeat the previous analysis.

(f) What is the Fourier transform of $\dfrac{1}{\pi(1+t^2)}$?

(g) Use the time shift property to deduce the Fourier transform of $\dfrac{1}{\pi(1+(t-\mu)^2)}$

(h) Find the Fourier transform of the pdf of $\mathbb{Z} = \mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n$.

(i) Show that $\mathbb{W} = \dfrac{\mathbb{X}_1 + \mathbb{X}_2 + \cdots + \mathbb{X}_n}{n}$ has the same Cauchy pdf as the $\mathbb{X}_i$. Thus, if your measurements have median value μ but have a Cauchy distribution, then the average of n measurements has the *same* pdf as each individual measurement, and is thus no more likely to be close to the median than the individual measurement.

Note that Fourier transforms are found in probability theory under the name *characteristic functions* but are not discussed in the textbook.