University of Illinois **Fall 2009**

ECE 313: Problem Set 11
Poisson Processes; Function of a Random Variable; Decision-Making

Due:	Wednesday November 11 at 4 p.m.
Reading:	Ross, Chapter 5; Powerpoint Lecture Slides, Sets 25-29
Noncredit Exercises:	**Chapter 5:** Problems 37-41;
	Theoretical Exercises 18-21, 29, 30; Self-Test Problems 14, 16, 20

1. [**Conditional Probabilities in a Poisson Process**]
Consider a Poisson process with arrival rate λ. Given that M packets arrived in $(0,t]$, what is the conditional probability that exactly k packets arrived in $(0,\tau]$, where $\tau < t$ and $0 \leq k \leq M$? That is, find $P\{\mathbb{N}(0,\tau] = k \mid \mathbb{N}(0,t] = M\}$.

2. [**Random chords**]
It might help to read Example 3d in Chapter 5 of Ross before working this problem.

   Let the straight line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length $\mathbb{X}$ of the arc (measured clockwise around the circle) is a random variable uniformly distributed on $[0, 2\pi)$. Now consider the random chord AD whose length we denote by $\mathbb{L}$.

   (a) Find the probability that $\mathbb{L}$ is greater than the side of the equilateral triangle inscribed in the circle.

   (b) Express $\mathbb{L}$ as a function of the random variable $\mathbb{X}$, and find the pdf for $\mathbb{L}$.

3. [**Current through a semiconductor diode**]
The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathbb{V}$ with pdf

$$f_{\mathbb{V}}(u) = 0.5\exp(-|u|), \quad -\infty < u < \infty.$$

   So, the current $\mathbb{I} = I_0(\exp(\mathbb{V}) - 1)$ is also a continuous random variable.

   (a) What values can $\mathbb{I}$ take on?

   (b) Find the CDF of $\mathbb{I}$.

   (c) Find the pdf of $\mathbb{I}$.

4. [**An A/D converter**]
A signal $\mathbb{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathbb{Y}$ (where $\mathbb{Y} = \alpha$ if $\mathbb{X} > 0$ and $\mathbb{Y} = -\alpha$ if $\mathbb{X} \leq 0$) is used. Note that $\mathbb{Y}$ is a *discrete* random variable.

   (a) What is the pmf of $\mathbb{Y}$?

   (b) The *squared error* in representing $\mathbb{X}$ by $\mathbb{Y}$ is $\mathbb{Z} = \begin{cases} (\mathbb{X}-\alpha)^2, & \text{if } \mathbb{X} > 0, \\ (\mathbb{X}+\alpha)^2, & \text{if } \mathbb{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathbb{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathbb{Z}]$. Use LOTUS to easily calculate $\mathsf{E}[\mathbb{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathbb{Z}]$.

   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathbb{X}$ to the nearest integer $\mathbb{W}$ in the range -3 to $+3$. Thus, $\mathbb{W} = 3$ if $\mathbb{X} \geq 2.5$, $\mathbb{W} = 2$ if $1.5 \leq \mathbb{X} < 2.5$, $\mathbb{W} = 1$ if $0.5 \leq \mathbb{X} < 1.5$, $\cdots$, $\mathbb{W} = -3$ if $\mathbb{X} < -2.5$. Note that $\mathbb{W}$ is also a discrete random variable. Find the pmf of $\mathbb{W}$.

(d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathbb{W}$. Suppose that the output is $(\mathbb{Z}_2, \mathbb{Z}_1, \mathbb{Z}_0)$. What is the pmf of $\mathbb{Z}_2$? the pmf of $\mathbb{Z}_1$? the pmf of $\mathbb{Z}_0$? Note that $(1,0,0)$ which represents -4 is not one of the possible outputs from this A/D converter.

5. **[Decision Making]**
Consider the following decision-making problem.
If hypothesis H_0 is true, the continuous random variable $\mathbb{X} \sim U(-2,2)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is $f_1(u) = \begin{cases} \frac{1}{4}(2-|u|), & |u|<2, \\ 0, & \text{otherwise.} \end{cases}$

(a) The *maximum-likelihood* decision rule can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$. Specify whether x denotes 0 or 1, and find the values of η, the probability of false alarm P_{FA}, and the probability of missed detection P_{MD}.

(b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

(c) The MAP decision rule (also known as the minimum-error-probability or Bayesian decision rule) can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether x denotes 0 or 1, and find the values of ξ and the error probability $P(E)$.

(d) For what range (if any) of values of π_0, does the MAP decision rule always choose hypothesis H_0?

(e) For what range (if any) of values of π_0, does the MAP decision rule always choose hypothesis H_1?