University of Illinois **Fall 2009**

ECE 313: Problem Set 10
Exponential and Gaussian Random Variables; Poisson Processes

Due:	Wednesday November 4 at 4 p.m.
Reading:	Ross, Chapter 5; Powerpoint Lecture Slides, Sets 24-27
	Qfunction: Note available on the COMPASS web page
Noncredit Exercises:	**Chapter 5:** Problems 7, 11, 12 15-24, 32, 33;
	Theoretical Exercises 2, 5, 8, 9; Self-Test Problems 8-11

1. **[Reliability function of a triple-modular-redundancy (TMR) system]**
Consider a triple modular redundancy (TMR) system (cf. Lecture 19 of the Powerpoint slides) with a perfect majority gate. The three modules have lifetimes denoted by $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ and fail independently of each other, that is, as in Problem 5 of Problem Set 9, the events $\{\mathbb{X}_1 > t_1\}$, $\{\mathbb{X}_2 > t_2\}$, and $\{\mathbb{X}_3 > t_3\}$ are independent events for all choices of t_1, t_2, and t_3. The modules are identical and hence we assume that they have identical reliability functions: $P\{\mathbb{X}_i > t\} = R(t)$ for $i = 1, 2, 3$. Let $\mathbb{Y}$ denote the lifetime of the TMR system.

   (a) Express the event $\{\mathbb{Y} > t\}$ in terms of unions, intersections and complements of the events $\{\mathbb{X}_1 > t\}, \{\mathbb{X}_2 > t\}, \{\mathbb{X}_3 > t\}$ and use this result to express the reliability function $R_{\mathbb{Y}}(t)$ of the TMR system in terms of $R(t)$.

   Henceforth, assume that $\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3$ are exponential random variables with parameter λ.

   (b) Express $R_{\mathbb{Y}}(t)$ as a function of λ and t and use this to find $\mathsf{E}[\mathbb{Y}]$, the *average lifetime* of the TMR system and the *median lifetime* of the TMR system. Compare your answers to the average and median lifetimes of a single module.

   (c) Now suppose that $\lambda = -\ln 0.999$. What are the numerical values of $P\{\mathbb{X}_1 > 1\}$ and $P\{\mathbb{Y} > 1\}$?

   (d) We hope you found in part (c) that $P\{\mathbb{X}_1 > 1\} = 0.999$ and so a single module works with 99.9% reliability for at least one unit of time. What is the largest value of T for which $P\{\mathbb{Y} > T\} \geq 0.999$? How does the TMR system compare to a single module in terms of providing 99.9% reliability over long periods of time?

   (e) Show that $R_{\mathbb{Y}}(t) = R(t)$ only for $t = 0$, $\lambda^{-1}\ln 2$, and ∞, and that $R_{\mathbb{Y}}(t) > R(t)$ for $0 < t < \lambda^{-1}\ln 2$.

2. **[A bound on the complementary Gaussian CDF]**
Let $\mathbb{X}$ denote a unit Gaussian random variable. Its CDF is $\Phi(u)$.

   (a) What is the derivative of $\exp(-u^2/2)$? Use this result to compute $\mathsf{E}[|\mathbb{X}|]$.

   (b) $Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{u^2}{2}\right) du = P\{\mathbb{X} > x\}$ is called the *complementary* CDF.
   A useful bound is $Q(x) \leq \frac{1}{2}\exp(-x^2/2)$ for $x \geq 0$. Derive this bound by first proving that $t^2 - x^2 > (t-x)^2$ for $t > x > 0$ and then applying this to

   $$\exp(x^2/2)Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{t^2 - x^2}{2}\right) dt.$$

3. **[Working with a table of the unit Gaussian distribution function]**
The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

   (a) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

(b) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

4. **[DeMoivre-Laplace approximation to central term of binomial distribution]**
Let n be a positive *even* integer, and let $\mathbb{X}$ be a binomial random variable with parameters $(n, 0.5)$. This problem focuses on $P\{\mathbb{X} = \frac{n}{2}\}$. The continuity correction for approximating the binomial distribution by the normal distribution begins by writing this same probability as $P\{\frac{n-1}{2} \leq \mathbb{X} \leq \frac{n+1}{2}\}$.

   (a) Using the continuity correction, find the normal approximation to $P\{\mathbb{X} = \frac{n}{2}\}$. Your answer should involve n and the standard normal CDF $\Phi(x)$.

   (b) Find the constant c such that $\sqrt{n}P\{\mathbb{X} = \frac{n}{2}\} \to c$ as $n \to \infty$, assuming you can replace $P\{\mathbb{X} = \frac{n}{2}\}$ by its normal approximation found in part (a). This suggests that $P\{\mathbb{X} = \frac{n}{2}\} \approx \frac{c}{\sqrt{n}}$ for large n. (Hint: Since $\Phi(x)$ is differentiable for all x, then $\frac{\Phi(h)-\Phi(0)}{h} \to \frac{d}{dx}\Phi(x)\big|_{x=0} = \Phi'(0)$ as $h \to 0$.)

   (c) For $n = 30$, compute the exact value of $P\{\mathbb{X} = \frac{n}{2}\}$, the approximation found in part (a), and the approximation found in part (b).

5. **[Poisson process probabilities]**
Consider a Poisson process with arrival rate $\lambda > 0$.

   (a) Find the probability there is exactly one arrival in each of the intervals (0,1], (1,2], and (2,3].

   (b) Find the probability that there are two arrivals in the interval $(0, 2]$ and two arrivals in the interval $(1, 3]$. (Hint: The occurrence of the event in (a) implies the occurrence of the event in (b). What other ways can the event in (b) occur?)

   (c) Find the probability that there are two arrivals in the interval (1,2], given that there are two arrivals in the interval (0,2] and two arrivals in the the interval (1,3].