University of Illinois **Fall 2009**

ECE 313: Problem Set 3
Discrete Random Variables: pmf, expectation, LOTUS, and variance

Due:	Wednesday September 16 at 4 p.m..
Reading:	**Ross, Chapter 4**
Noncredit Exercises:	DO NOT turn these in. **Chapter 4:** Problems 2, 7, 13, 28, 35, 39, 40-43 Theoretical Exercises 11, 13, 15; Self-Test Problems 1-10.

1. [**Rolling a die till something different happens**]
Consider the die-rolling experiment described in Problem 3 of Problem Set 2 in which a die is rolled repeatedly until an outcome *different* from the outcome of the first roll is observed.

   (a) Let $\mathbb{X}$ denote the number of rolls of the die on a trial. What values can $\mathbb{X}$ take on? What is the probability mass function (pmf) of $\mathbb{X}$? What is the probability that $\mathbb{X}$ is an even number?

   (b) A random variable $\mathbb{Y}$ is defined on this experiment as follows:

$$\mathbb{Y} = \begin{cases} i, & \text{if outcome of the fourth roll is } i, \\ 0, & \text{if experiment ended in two or three rolls.} \end{cases}$$

   What is the pmf of $\mathbb{Y}$? Hint: Find $p_{\mathbb{Y}}(0)$ first and then argue that $p_{\mathbb{Y}}(i) = (1-p_{\mathbb{Y}}(0))/6, 1 \leq i \leq 6$.

2. [**The Game of Chuck-A-Luck**]
In the game of Chuck-A-Luck played at fairs and carnivals in the MidWest, bets are placed on numbers 1, 2, 3, 4, 5, 6, and then three fair dice are rolled. If the number chosen does not show up on any of the three dice, the bettor loses his stake. Otherwise, the dealer pays the bettor one or two or three times the amount staked according as the number chosen shows up on one or two or all three of the dice. Of course, the amount staked is also returned to the bettor but is *not* counted as part of the *winnings* from this game. Let $\mathbb{X}$ denote the winnings in this game for a \$6 bet, and remember that negative values of $\mathbb{X}$ correspond to losses.

   (a) What are the values taken on by $\mathbb{X}$?

   (b) What is the pmf of $\mathbb{X}$?

   (c) What is the value of $\mathsf{E}[\mathbb{X}]$?

   (d) A player splits his \$6 bet and wagers \$1 on each of the six numbers. Let $\mathbb{Y}$ denote the winnings of this player. Repeat parts (a)-(c) for $\mathbb{Y}$. Does the splitting strategy improve the average winnings in this game?

3. [**A different calculation for expected values**]
In this problem, $\mathbb{X}$ denotes a discrete random variable taking on nonnegative integer values only.

   (a) Suppose $\mathbb{X}$ takes on values 1, 2, or 3 only. Show that $\mathsf{E}[\mathbb{X}] = \sum_{k=0}^{2} P\{\mathbb{X} > k\}$.

   Hint: Write out $\mathsf{E}[\mathbb{X}]$ as a sum of 6 values of $p_{\mathbb{X}}(u)$ and re-arrange the sum.

   (b) Show that $\mathsf{E}[\mathbb{X}] = \sum_{k=0}^{\infty} P\{\mathbb{X} > k\}$ for a nonnegative integer-valued random variable $\mathbb{X}$.

   Hint: re-arrange a sum again (but more systematically than you did in part (a)).

4. **[Is that your final answer?]**
Consider a quiz game in which you are given two Questions. You think you can answer Question 1 correctly with probability 0.8 and Question 2 correctly with probability 0.5. You win \$100 for answering Question 1 correctly and \$200 for answering Question 2 correctly. (An incorrect answer means you win nothing.) You can choose which Question you want to answer. Let $\mathbb{X}$ denote the amount that you win.

   (a) Calculate the pmf and expectation of $\mathbb{X}$ for the two cases:
   (i) You choose Question 1. (ii) You choose Question 2.
   Which Question should you choose to answer to maximize $\mathsf{E}[\mathbb{X}]$?

   In a variation of the quiz game, you answer your chosen Question, *and if you answer correctly, then and only then do you get to answer the other Question* and possibly win the prize for answering the other Question correctly. If your chosen Question is answered incorrectly, you don't get to answer the other Question and you go back to your seat with winnings of \$0. Let $\mathbb{Y}$ denote your winnings in this game.

   (b) Calculate the pmf and expectation of $\mathbb{Y}$ for the two cases:
   (i) You choose Question 1 to answer first. (ii) You choose Question 2 to answer first.
   Which Question should you choose to answer to maximize $\mathsf{E}[\mathbb{Y}]$?

   (c) More generally, let p_1 and p_2 denote the probabilities of correctly answering Questions 1 and 2 respectively, and v_1 and v_2 the prizes for correctly answering Questions 1 and 2 respectively. Show that the optimal strategy for the second game (with winnings $\mathbb{Y}$) is to answer Question 1 first if

$$\frac{p_1 v_1}{1-p_1} > \frac{p_2 v_2}{1-p_2}.$$

5. **[Finding the mean and the variance]**
Consider the sample space defined in Problem 5 of Problem Set 2 and suppose that the random variable $\mathbb{X}$ has value n whenever the outcome is the number n so that $p_{\mathbb{X}}(n) = \dfrac{(\log_e 2)^n}{2(n!)}$ for $n \geq 0$.

   (a) Find $\mathsf{E}[\mathbb{X}]$, the expected value (or mean) of the random variable $\mathbb{X}$.

   (b) Use LOTUS to find $\mathsf{E}[\mathbb{X}(\mathbb{X}-1)]$, and use this result to deduce the values of $\mathsf{E}[\mathbb{X}^2]$, the mean-square value of $\mathbb{X}$, and $\mathsf{var}[\mathbb{X}]$, the variance of the random variable $\mathbb{X}$.

6. **[If at first you don't succeed, (let the search engine) try, try again – American Proverb]**
Packet transmission on the Internet uses a communications protocol known as ARQ (Automatic Repeat reQuest) in which the receiver (automatically) asks the transmitter to repeat a packet that has been received in error. If the packet is received in error again, another re-transmission is requested, and the packet transmitted again. This process continues until the packet finally is received correctly. Suppose that a packet is received correctly with probability 0.9. Let $\mathbb{X}$ denote the total number of times that a packet is transmitted.

   (a) What values does $\mathbb{X}$ take on, and what is the pmf of $\mathbb{X}$?

   (b) What is the expected value of $\mathbb{X}$? Hint: see Solution to Problem 2(d) of Problem Set 0.

   (c) In practice, repeats are requested a fixed number of times, say $K-1$ times, and if the packet is received incorrectly all K times, it is considered to be lost, and no further re-transmissions are requested. The *throughput* of the channel is $\mathbb{T}$ which has value $1/\mathbb{X}$ if the packet was received correctly after $\mathbb{X} \leq K$ transmissions, while if all K packet transmissions are received incorrectly, then $\mathbb{T}$ has value 0. Find $P\{\text{packet loss}\}$, and use LOTUS to find the *average throughput* $\mathsf{E}[\mathbb{T}]$.