University of Illinois **Fall 2009**

ECE 313: Problem Set 1
Sets, Events, Axioms of Probability and Their Consequences

Due:	Wednesday September 2 at 4 p.m..
Reading:	*Ross* Chapter 1, Sections 1-4; Chapter 2, Sections 1-5
	Powerpoint Lecture Slides, Sets 1-6
Noncredit Exercises:	**Chapter 1:** Problems 1-5, 7, 9;
	Theoretical Exercises 4, 8, 13; Self-Test Problems 1-15.
	Chapter 2: Problems 3, 4, 9, 10, 11-14;
	Theoretical Exercises 1-3, 6, 7, 10, 11, 12, 16, 19, 20; Self-Test Problems 1-8

Yes, the reading and noncredit exercises are the same as in Problem Set 0.

1. **[Subsets of a finite set]**
   Let Ω denote a finite set containing the n elements $\omega_1, \omega_2, \ldots, \omega_n$. The *cardinality* (more informally, the *size*) of a subset $A \subset \Omega$ is the number of elements in A, and is denoted as $|A|$.

   (a) Let $n = 4$. List *all* the subsets of Ω in increasing order of size. How many subsets are there? How many of these subsets are *non-empty* subsets?

   (b) If you listed only 14 or 15 subsets in part (a), please re-do part (a), and this time, include the *empty set* $\emptyset$ and/or Ω as subsets of Ω.

   (c) In your answer to part (a) or (b), verify that for each k, $0 \leq k \leq 4$, the *total number* of subsets of size k is the same as the *total number* of subsets of size $4-k$. Now explain why for n in general, the total number of subsets of size k is the same as the total number of subsets of size $n-k$.

   (d) Each subset A corresponds to a n-bit vector $(x_1, x_2, \ldots, x_n)$ where $x_i = 1$ if $\omega_i \in A$ and $x_i = 0$ if $\omega_i \notin A$. Writing $A \leftrightarrow (x_1, x_2, \ldots, x_n)$ emphasizes the *one-to-one correspondence*: each subset defines a unique n-bit vector, and each n-bit vector defines a unique subset, e.g. with $n = 4$, we have that $\{\omega_1, \omega_3\} \leftrightarrow (1, 0, 1, 0)$.

      i. What n-bit vectors correspond to Ω and to $\emptyset$? What n-bit vector corresponds to A^c?

      ii. If $A \leftrightarrow (x_1, x_2, \ldots, x_n)$, $B \leftrightarrow (y_1, y_2, \ldots, y_n)$, $(A \cup B) \leftrightarrow (z_1, z_2, \ldots, z_n)$ and $(A \cap B) \leftrightarrow (w_1, w_2, \ldots, w_n)$, express the z_i's and w_i's in terms of the x_i's and y_i's. Hint: you may need the logical operators $\vee$ and $\wedge$ that you may have encountered in ECE 290.

      iii. How many different n-bit vectors are there? How many different subsets are there of Ω?

      iv. "They correspond to the nonempty subsets of Ω" Respond as if you are on Jeopardy$^{\text{TM}}$: What is the question to which this statement is the answer? How many subsets of Ω are non-empty?

   (e) One definition of $\binom{n}{k}$ is the total number of subsets of size k of a set Ω of size n.

      i. Compute the numerical values of $\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}$, and $\binom{4}{0}$. Do the numbers match up with your answers to part (a) or (b)?

      ii. The *total number* of subsets whose *size* is an *even* number can be expressed as $\binom{n}{0} + \binom{n}{2} + \cdots$ What is the last term in this series? Be careful to distinguish between the cases: n is odd, and n is even.

      iii. Write a similar expression for the *total number* of subsets whose *size* is an *odd* number while continuing to be careful to distinguish between the cases: n is odd, and n is even.

      iv. Show that there are exactly 2^{n-1} subsets whose *size* is an *even* number, and exactly 2^{n-1} subsets whose *size* is an *odd* number.
      Hint: expand $(1-x)^n$ using the binomial theorem and then set $x = 1$.

2. [**A problem on sampling without replacement**]
A bag contains n pairs of shoes in distinct styles and sizes. You pick two shoes at random from the bag. Note that this is sampling *without* replacement.

   (a) What is the probability that you get a pair of shoes?

   (b) What is the probability of getting one left shoe and one right shoe?

   Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.

   (c) What is the probability that you have a pair of shoes among the three that you have picked?

   (d) What is the probability that you picked at least one left shoe and at least one right shoe?

3. [**Unions and Intersections**]
An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

   (a) Let A denote the event that the least significant bit (LSB) is a ONE. What is $P(A)$?

   (b) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B)$?

   (c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of the two events A and B occurs, i.e. what is $P(A \oplus B)$?

4. [**New NCAA tournament rules: the Final Five instead of the Final Four!**]
Five basketball teams (call them A, B, C, D, and E) play a round-robin tournament in which each team plays each of the other four exactly once.

   (a) How many basketball games are there in this tournament? More generally, how many games would there be in an n-team round-robin tournament?

   (b) Now suppose that each game is equally likely to end in a win for either team, and so the 2^M possible outcomes of the M games all have the same probability 2^{-M}.

      i. What is the probability that some team wins all four of its games?

      ii. What is the probability that some team loses all four of its games?

      iii. What is the probability that one team wins all four of its games *and* another loses all four of its games?

      iv. What is the probability that one team wins all four of its games *and* another loses all four of its games *and* the remaining teams have identical 2-2 records?

5. [**Drill in working with subsets**]
Find $P(A \cup (B^c \cup C^c)^c)$ in each of the following four cases:

   (a) A, B, and C are disjoint (mutually exclusive) events and $P(A) = 1/3$.

   (b) $P(A) = 2P(B \cap C) = 4P(A \cap B \cap C) = 1/2$.

   (c) $P(A) = 1/2$, $P(B \cap C) = 1/3$, and $P(A \cap C) = 0$.

   (d) $P(A^c \cap (B^c \cup C^c)) = 0.6$.