ECE 313: Problem Set 6: Solutions
Bayes' Formula and Midterm Review Problems

1. **[Conditional Probabilities]**

   (a) Yes, A gives positive information about B. Note that by the definition of conditional probability, $P(A \mid B) = P(A \cap B)/P(B) > P(A)$ is equivalent to $P(A \cap B) > P(A)P(B)$, and so it is also true that $P(B \mid A) = P(A \cap B)/P(A) > P(B)$.

   (b) Yes, B^c give *negative information* about A, that is, it *is* true that $P(A \mid B^c) < P(A)$. Notice from part (a) that $P(A \cap B) > P(A)P(B)$ and so

   $$P(A) = P(A \cap B) + P(A \cap B^c) > P(A)P(B) + P(A \cap B^c).$$

   Therefore, it must be that $P(A)(1 - P(B)) = P(A)P(B^c)$ is larger than $P(A \cap B^c)$, and so

   $$P(A) > \frac{P(A \cap B^c)}{P(B^c)} = P(A \mid B^c),$$

   that is, if B gives positive information about A, then B^c gives negative information about A.

   (c) B^c gives *positive information* about A^c. Since part (b) gives us that $P(A \mid B^c) < P(A)$, it follows that $P(A^c \mid B^c) = 1 - P(A \mid B^c) > 1 - P(A) = P(A^c)$ and so $P(A^c \mid B^c) > P(A^c)$.

2. **[Hindsight]**

   Let an experiment consist of picking a bulb at random (say from a box containing bulbs manufactured by Transylvania and Eastinghouse) and using it till the bulb burns out. Let T denote the event that the bulb picked was made by Transylvania Corp. and $E = T^c$ the event that the bulb was made by Eastinghouse Corp. We are told that $P(E) = P(T) = 0.5$, that is, we can suppose that the box contains equal numbers of bulbs made by the two companies. If $\mathbb{X}$ denotes the lifetime (rounded up to the next full hour) of the bulb, then we are also told that $P\{\mathbb{X} = n \mid T\} = p_1(n)$ and $P\{\mathbb{X} = n \mid E\} = p_2(n)$.

   (a) By the law of total probability (LOTP),

   $$\begin{aligned} P\{\mathbb{X} = M\} &= P\{\mathbb{X} = M \mid T\}P(T) + P\{\mathbb{X} = M \mid E\}P(E) \\ &= p_1(M)P(T) + p_2(M)P(E) = \frac{p_1(M) + p_2(M)}{2}. \end{aligned}$$

   Be sure that you understand that you cannot just add $p_1(M)$ and $p_2(M)$ to find $P\{\mathbb{X} = M\}$, nor can you *in general* simply compute the arithmetic average of $p_1(M)$ and $p_2(M)$; you have to compute a *weighted sum* of $p_1(M)$ and $p_2(M)$ as specified by the law of total probability. In *this case*, the LOTP calculation turns out to be the arithmetic average because it so happens that $P(E) = P(T) = 0.5$, that is, the weights are $\frac{1}{2}$ for both

   (b) Note that in accordance with common practice, the word *condiitonal* is missing from the problem statement, but what is really being asked for is $P\{T \mid \mathbb{X} = M\}$. So, this is a Bayes' formula calculation:

   $$P\{T \mid \mathbb{X} = M\} = \frac{P\{\mathbb{X} = M \mid T\}P(T)}{P\{\mathbb{X} = M\}} = \frac{p_1(M)\frac{1}{2}}{(p_1(M) + p_2(M))/2} = \frac{p_1(M)}{p_1(M) + p_2(M)}.$$

   (c) This is also a Bayes' formula calculation except that the event of interest is $\{\mathbb{X} > M\}$ whose probability is

   $$P\{\mathbb{X} > M\} = \sum_{n=M+1}^{\infty} P\{\mathbb{X} = n\} = 1 - \sum_{n=1}^{M} P\{\mathbb{X} = n\} = 1 - \sum_{n=1}^{M} \frac{p_1(n) + p_2(n)}{2}.$$

We thus have

$$P\{T \mid \mathbb{X} > M\} = \frac{P\{\mathbb{X} > M \mid T\}P(T)}{P\{\mathbb{X} > M\}} = \frac{\left(1 - \sum_{n=1}^{M} p_1(n)\right)\frac{1}{2}}{1 - \sum_{n=1}^{M} \frac{p_1(n)+p_2(n)}{2}}$$
$$= \frac{\left(1 - \sum_{n=1}^{M} p_1(n)\right)}{\left(1 - \sum_{n=1}^{M} p_1(n)\right) + \left(1 - \sum_{n=1}^{M} p_2(n)\right)}.$$

3. **[A woman is twice as good as a man!]**

Let us define the random variable of the face that Bob's roll of dice shows by b. Let B denote the event that Bob wins the game. Then, since Alice can lose i different ways if $b = i$, it is clear that for $i = 1, 2, 3, 4, 5$, and 6, $P(B \mid b = i) = i/6$. By the law of total probability,

$$P(B) = \sum_{i=1}^{6} P(B \mid b = i)P(b = i) = \frac{1+2+3+4+5+6}{6} \times \frac{1}{6} = \frac{21}{36} = \frac{7}{12}.$$

Next, note that *both* of Alice's rolls must result in numbers *larger* than Bob's number in order for Alice to win. Let A denote the probability that Alice wins the game. If Bob rolls an i, then the (conditional) probability that Alice beats him twice is $P(A \mid b = i) = \frac{(6-i)^2}{36}$. Again by the law of total probability,

$$P(A) = \sum_{i=1}^{6} P(A \mid b = i)P(b = i) = \frac{5^2 + 4^2 + 3^2 + 2^1 + 1^2 + 0^2}{36} \times \frac{1}{6} = \frac{55}{216}.$$

Let D denote the event of a draw, which occurs whenever Alice's first roll beats Bob, but the second does not. Thus, $P(D \mid b = i) = \frac{(6-i)i}{36}$. Once again, the law of total probability rides to the rescue giving us

$$P(D) = \sum_{i=1}^{6} P(D \mid b = i)P(b = i) = \frac{5 \times 1 + 4 \times 2 + 3 \times 3 + 2 \times 4 + 1 \times 5 + 0 \times 6}{36} \times \frac{1}{6} = \frac{35}{216}.$$

Sure, the three probabilities add up to 1 (either Bob wins, or Alice wins or there is a tie): no surprise here, you can verify this directly by summing up the three probabilities.

4. **[Events and K-maps]**

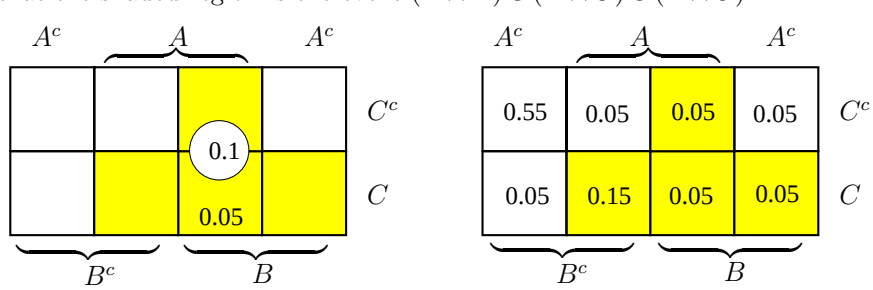
   (a) Events are mutually exclusive if they cannot occur at the same time. None of these events are mutually exclusive.

   (b) Independent pairs of events are $\{A, B\}, \{A, C\}, \{A, D\}$.

   (c) D contains both C and B. B contains C.

5. **[The noisy cereal problem]**

   (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event $(A \cap B) \cup (B \cap C) \cup (A \cap C)$.



(b) Since $A \cap B$ is the disjoint union of $A \cap B \cap C$ and $A \cap B \cap C^c$, we get that $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$ giving that $P(A \cap B \cap C^c) = 0.05$. Since $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$ while $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$, we readily obtain that $P(AB^cC) = 0.15$ and $P(A^cBC) = 0.05$.
Since $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$ we have that
$P(A^cB^c) = 1 - P(A \cup B) = 0.4$. Since $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$, we get that $P(\text{cereal snaps, crackles, and pops}) = P(A^c \cap B^c \cap C^c) = 0.55$. See the figure on the right.

(c) $P(\text{the sample fails } \textit{only} \text{ the snap test}) = P(AB^cC^c) = 0.05$.
$P(\text{the sample fails } \textit{only} \text{ the crackle test}) = P(A^cBC^c) = 0.05$.
$P(\text{the sample fails } \textit{only} \text{ the pop test}) = P(A^cB^cC) = 0.05$.

6. **[Optimizing Flight Durations]**

(a) Alice arrives on schedule if there is no delay on the SC leg *and* the CN leg. Both these events are independent and have probability $1 - p$ each. So the probability that Alice arrives on schedule in N is $(1 - p)^2$ Alice arrives exactly an hour behind schedule in N when either:

- the SC leg is delayed by one hour and the CN leg is not delayed, or
- the SC leg is not delayed and the CN leg is delayed by one hour.

Both these events are disjoint and have probability $p(1 - p)$. So the probability that Alice arrives in N an hour behind schedule is $2p(1 - p)$.

Alice arrives exactly two hours behind schedule in N when both the SC and CN legs are delayed by one hour each. So, the probability that Alice arrives in N two hours behind schedule is p^2.

Let us denote the delay (in number of hours) for the SC leg by the random variable d_{SC} and the delay for the CN leg by the random variable d_{CN}. Both these random variables are *Bernoulli*, i.e.,

$$\begin{aligned} P(d_{SC} = 0) &= P(d_{CN} = 0) = (1 - p) \\ P(d_{SC} = 1) &= P(d_{CN} = 1) = p. \end{aligned}$$

The average of both d_{SC} and d_{CN} is p. We are interested in the average of the total delay (which is equal to $d_{SC} + d_{CN}$). By the linearity property of expectation, this is just $2p$.

(b) The random variables d_{SD} and d_{DN} (representing the delays on the SD and DN legs respectively) are also sort of Bernoulli:

$$\begin{aligned} P(d_{SD} = 0) &= P(d_{DN} = 0) = (1 - q) \\ P(d_{SD} = 2) &= P(d_{DN} = 2) = q. \end{aligned}$$

So, the answers to this question are quite similar to the previous part: $(1 - q)^2, 2q(1 - q), q^2$ and $4q$, respectively.

(c) We can use the theorem of total probability to answer this question: we know the average delay conditioned on the two routes (Chicago and Denver) to be $2p$ and $4q$, respectively. Either of these two routes is taken with equal probability, so the average delay is

$$\frac{1}{2} \cdot 2p + \frac{1}{2} \cdot 4q = p + 2q.$$

(d) Let us denote the total delay by the random variable d. We are asked to calculate

$$P(\text{SCN route is taken} | d = 2).$$

We do know that

$$P(d = 2 | \text{SCN route is taken}) = p^2,$$

so this problem is ripe for an application of the Bayes rule:

$$P(\text{SCN route is taken}|d=2) = \frac{P(d=2|\text{SCN route is taken}) \cdot P(\text{SCN route is taken})}{P(d=2)}.$$

We know the values of the two probabilities in the numerator to be p^2 and 0.5 respectively. We can use the theorem of total probability to calculate the denominator:

$$\begin{aligned} P(d=2) &= P(\text{SCN route is taken}) \cdot P(d=2|\text{SCN route is taken}) + \\ &\qquad P(\text{SDN route is taken}) \cdot P(d=2|\text{SDN route is taken}) \\ &= \frac{1}{2} \cdot p^2 + \frac{1}{2} 2q(1-q). \end{aligned}$$

Putting these together, we conclude that

$$P(\text{SCN route is taken}|d=2) = \frac{p^2}{p^2 + 2q(1-q)}.$$

(e) The probability that Alice arrives in N less than two hours behind schedule depends on the route she takes. In the two cases, we have

$$\begin{aligned} P(d<2|\text{SCN route is taken}) &= \sum_{k=0,1} P(d=k|\text{SCN route is taken}), \\ &= 1-p^2, \end{aligned}$$

and

$$\begin{aligned} P(d<2|\text{SDN route is taken}) &= P(d=0|\text{SCN route is taken}), \\ &= (1-q)^2. \end{aligned}$$

So the SCN route is better than the SDN route whenever

$$1-p^2 > (1-q)^2.$$

(f) Let X denote the number of trips Alice makes on the SCN route until she experiences a 2 hour delay. We know that X is a *geometric* random variable:

$$P(X>k) = (1-p^2)^k.$$

(This is because the first k trips all must have a delay of 0 or 1 hour, the probability of which is $1-p^2$ for each trip.) We know from the lectures that the average of the geometric random variable X is

$$\frac{1}{p^2}.$$