University of Illinois Fall 2009

ECE 313: Final Exam

Tuesday, December 15, 2009, 8:00 a.m. — 11:00 a.m.
100 Materials Science and Engineering Building

1. (a) A, B, and C are events such that $0 < P(A) < 1$, $0 < P(B) < 1$, and $0 < P(C) < 1$.
   - $P(A \cup B) = P(A^c \cup B^c) - P(A^c) - P(B^c) + 1$ is a TRUE statement.
     Note that the right side is $-P(A^c \cap B^c) + 1$ and the result follows from DeMorgan's theorem.
   - $P(AB) \geq P(A) + P(B) - 1$ is a TRUE statement.
     Transposition gives $P(A) + P(B) - P(AB) = P(A \cup B) \leq 1$ which obviously is true.
   - $P(A^c \mid B)P(B) + P(A^c \mid B^c)P(B^c) = P(A^c)$ is a TRUE statement.
     It is just the law of total probability applied to $P(A^c)$.
   - $P(A^c \mid B)P(B) + P(A \mid B)P(B) = P(B)$ is a TRUE statement.
     Note that $P(B)$ is a common factor on the left side and $P(A^c \mid B) + P(A \mid B) = 1$,
   - $P(A^c \mid B)P(B) + P(A \mid B^c)P(B^c) = P(A \cup B) - P(AB)$ is a TRUE statement.
     Both sides equal $P(A \oplus B)$.
   - $P(B \mid A) = P(A \mid B)P(A)/P(B)$ is a FALSE statement.
     The similar-looking $P(B \mid A) = P(A \mid B)P(B)/P(A)$ is, of course, just Bayes' formula.
   - "If A and B are *mutually exclusive* events, then they are *independent* events" is a FALSE statement.
     Mutually exclusive events are independent only in the trivial case when at least one of the events has zero probability.
   - "If A, B, and C are *independent* events, then $P(ABC) = P(A)P(B)P(C)$" is a TRUE statement.
     The condition $P(ABC) = P(A)P(B)P(C)$ is one of four conditions that must hold for A, B, and C to be called independent events.
   - "If $P(ABC) = P(A)P(B)P(C)$, then A, B, and C are *independent*" is a FALSE statement.
     It is also necessary that $P(AB) = P(A)P(B)$, $P(AC) = P(A)P(C)$, and $P(BC) = P(B)P(C)$ hold in order for A, B, and C to be independent events.

   (b) $f_{\mathbb{X}}(u)$ is an even function and $\mathsf{var}(\mathbb{X}) = 4$.
   - $F_{\mathbb{X}}(u) = F_{\mathbb{X}}(-u)$ for all $u, -\infty < u < \infty$ is a FALSE statement.
     In fact, $F_{\mathbb{X}}(u) = 1 - F_{\mathbb{X}}(-u)$ for all $u, -\infty < u < \infty$.
   - $\mathsf{E}[\mathbb{X}^2] = 4$ is a TRUE statement. $\mathsf{E}[\mathbb{X}^2] = \mathsf{var}(\mathbb{X}) + (\mathsf{E}[\mathbb{X}])^2 = \mathsf{var}(\mathbb{X}) = 4$.
   - $\mathsf{E}[|\mathbb{X}|] = 2$ is a FALSE statement. If $\mathbb{X} \sim \mathcal{N}(0, 4)$, then $\mathsf{E}[|\mathbb{X}|] = 2\sqrt{\frac{2}{\pi}} \neq 2$.
   - $P\{\mathbb{X} > u\} = F_{\mathbb{X}}(-u)$ for all $u, \; -\infty < u < \infty$ is a TRUE statement.
     Note that $P\{\mathbb{X} > u\} = 1 - F_{\mathbb{X}}(u) = F_{\mathbb{X}}(-u)$ by symmetry of the pdf.
   - $P\{\mathbb{X} > u\} \leq 2u^{-2}$ is a TRUE statement.
     Since the pdf is symmetric about 0, $P\{\mathbb{X} > u\} = \frac{1}{2}P\{|\mathbb{X}| > u\}$. But the Chebyshev inequality gives that for any $u > 0$, $P\{|\mathbb{X}| > u\} \leq \frac{4}{u^2}$, and the result follows.
   - $P\{\mathbb{X}(\mathbb{X} - 1) < 2\} = P\{\mathbb{X}(\mathbb{X} + 1) < 2\}$ is a TRUE statement.
     Note that $\mathbb{X}(\mathbb{X} - 1) < 2$ if $-1 < \mathbb{X} < 2$ while $\mathbb{X}(\mathbb{X} + 1) < 2$ if $-2 < \mathbb{X} < 1$. But, by symmetry of the pdf, $P\{-1 < \mathbb{X} < 2\} = P\{-2 < \mathbb{X} < 1\}$.

   (c)
   - "The pdf of the sum $\mathbb{X} + \mathbb{Y}$ is $f_{\mathbb{X}+\mathbb{Y}} = f_{\mathbb{X}} \star f_{\mathbb{Y}}$" is a FALSE statement.
     Note that the statement would be true if $\mathbb{X}$ and $\mathbb{Y}$ were independent random variables.
   - "If $\mathbb{X}$ and $\mathbb{Y}$ are *independent* random variables, then $\mathbb{X}^2$ and $\mathbb{Y}^2$ are independent" is a TRUE statement.
     In fact, more generally, $g(\mathbb{X})$ and $h(\mathbb{Y})$ are independent random variables.
   - " If $\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated* random variables, then $\mathbb{X}^2$ and $\mathbb{Y}^2$ are uncorrelated" is a FALSE statement.
     The statement could be true in special cases but is not true in general.
   - "If $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$, then $\mathsf{cov}(\mathbb{X} + \mathbb{Y}, \mathbb{X} - \mathbb{Y}) = 0$" is a TRUE statement.
     $\mathsf{cov}(\mathbb{X}+\mathbb{Y}, \mathbb{X}-\mathbb{Y}) = \mathsf{var}(\mathbb{X}) - \mathsf{var}(\mathbb{Y})$ is easily shown, or just use the bilinearity of the covariance function.

- "If $\mathbb{X}$ and $\mathbb{Y}$ are *independent* random variables, then $\mathsf{var}(3\mathbb{X}+2\mathbb{Y}) = \mathsf{var}(-3\mathbb{X}+2\mathbb{Y})$" is a TRUE statement.
  The left side equals $9 \cdot \mathsf{var}(\mathbb{X}) + 4 \cdot \mathsf{var}(\mathbb{Y}) + 12 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y})$;
  the right side equals $9 \cdot \mathsf{var}(\mathbb{X}) + 4 \cdot \mathsf{var}(\mathbb{Y}) - 12 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y})$ but the covariance is zero.

2. (a) Tom goes home after three tosses if and only if the first three tosses result in HHH. This event has probability $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and Tom's wealth is \$8.

   (b) Tom goes home after five tosses and \$0 if and only if the first five tosses result in TTTTT. This event has probability $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.

   (c) Tom goes home after 5 tosses with \$8 if the outcomes of the tosses is any of THHHH, HTHHH, HHTHH. So the probability that Tom goes home after 5 tosses is

$$P\{\text{TTTTT, THHHH, HTHHH, HHTHH}\} = 4 \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{8}.$$

   (d) Since Tom tosses the coin at least 5 times, we know that he did not go home with \$8 after 3 tosses (which event has probability 1/8 as calculated in part (a)). It is straightforward to determine that Tom does not go home after 4 coin tosses. So, the probability of the conditioning event A — Tom tossed the coin at least 5 times — is $P(A) = 7/8 = 28/32$.

   We have already calculated that $P(\{\mathbb{X}=0\} \cap A) = \frac{1}{32}$ and $P(\{\mathbb{X}=8\} \cap A) = \frac{3}{32}$ in part (c). The other possibilities are that Tom's *wealth* is

   - \$2 (1 Head and 4 Tails) which has probability $P(\{\mathbb{X}=2\} \cap A) = \binom{5}{1} \cdot \left(\frac{1}{2}\right)^5 = 5 \cdot \left(\frac{1}{2}\right)^5 = \frac{5}{32}$;
   - \$4 (2 Heads and 3 Tails) which has probability $P(\{\mathbb{X}=4\} \cap A) = \binom{5}{2} \cdot \left(\frac{1}{2}\right)^5 = 10 \cdot \left(\frac{1}{2}\right)^5 = \frac{10}{32}$;
   - \$6 (3 Heads and 2 Tails in 5 tosses *except* for outcome HHHTT which cannot occur) which has probability $P(\{\mathbb{X}=6\} \cap A) = \left(\binom{5}{3} - 1\right) \cdot \left(\frac{1}{2}\right)^5 = 9 \cdot \left(\frac{1}{2}\right)^5 = \frac{9}{32}$.

   Since $p_{\mathbb{X}|A}(i \mid A) = \dfrac{P(\{\mathbb{X}=i\} \cap A)}{P(A)} = P(\{\mathbb{X}=i\} \cap A) \times \dfrac{32}{28}$, we get that

$$p_{\mathbb{X}|A}(0 \mid A) = \frac{1}{28}; \quad p_{\mathbb{X}|A}(2 \mid A) = \frac{5}{28}; \quad p_{\mathbb{X}|A}(4 \mid A) = \frac{10}{28}; \quad p_{\mathbb{X}|A}(6 \mid A) = \frac{9}{28}; \quad p_{\mathbb{X}|A}(8 \mid A) = \frac{3}{28}.$$

3. (a) Let $\mathbb{Y}$ denote the number of heads in 72 independent tosses of the biased coin. Then, $\mathbb{Y}$ is a *binomial* random variable with mean $np = 72 \times \frac{1}{3} = 24$ and variance $np(1-p) = 72 \times \frac{1}{3} \times \frac{2}{3} = 16$. Since Dick wins $\mathbb{Y}$ \$ and loses $72-\mathbb{Y}$ \$, his wealth is $\mathbb{X} = 100+\mathbb{Y}-(72-\mathbb{Y}) = 28+2\mathbb{Y}$ and so, we get that $\mathsf{E}[\mathbb{X}] = \mathsf{E}[28+2\mathbb{Y}] = 28+2\mathsf{E}[\mathbb{Y}] = 28+2\times 24 = 76$ and $\mathsf{var}(\mathbb{X}) = \mathsf{var}(28+2\mathbb{Y}) = 2^2\mathsf{var}(\mathbb{Y}) = 64$.

   (b) We have that $P\{\mathbb{X} \geq 64\} = P\{28 + 2\mathbb{Y} \geq 64\} = P\{\mathbb{Y} \geq 18\}$. By the DeMoivre-LaPlace approximation (which is a special case of the Central Limit Theorem),

$$P\{\mathbb{X} \geq 64\} = P\{\mathbb{Y} \geq 18\} \approx 1 - \Phi\left(\frac{18-24}{4}\right) = 1 - \Phi(-1.5) = \Phi(1.5) = 0.9332$$

   Purists might want to use the *continuity correction* to write

$$P\{\mathbb{X} \geq 64\} = P\{\mathbb{Y} \geq 18\} \approx 1 - \Phi\left(\frac{17.5-24}{4}\right) = 1 - \Phi(-1.625) \approx \Phi(1.62) = 0.9474$$

   or 0.9479 using linear interpolation between the values given in the table for $\Phi(1.62)$ and $\Phi(1.63)$.

4. (a) Since Harry accepts 3 of 12 signs, his chances of accepting a suggestion are $3/12 = 1/4$. The number of suggestions until Harry accepts one is thus a *geometric* random variable with parameter 1/4, and hence expected value 4. Therefore Harry rejects 3 suggestions on average.

   (b) Let A denote the event that Harry accepts the candidate. Then, we are given that $P(H) = \frac{1}{3}$; $P(N) = \frac{2}{3}$ and that $P(A \mid H) = \frac{3}{4}$ and $P(A \mid N) = \frac{3}{40}$. Hence, by the law of total probability,

$$P(A) = P(A \mid H)P(H) + P(A \mid N)P(N) = \frac{3}{4} \times \frac{1}{3} + \frac{3}{40} \times \frac{2}{3} = \frac{1}{4} + \frac{1}{20} = \frac{5+1}{20} = \frac{6}{20} = \frac{3}{10}.$$

Furthermore, using Bayes' formula, $P(H \mid A) = \dfrac{P(A \mid H)P(H)}{P(A)} = \dfrac{5/20}{6/20} = \dfrac{5}{6}$.
Note that all the numbers used are from the law of total probability calculation done above.

5. For any $n \geq 0$, $P\{\mathbb{X} > \mathbb{Y} \mid \mathbb{Y} = n\} = P\{\mathbb{X} > n \mid \mathbb{Y} = n\} = P\{\mathbb{X} > n\}$ since $\mathbb{X}$ and $\mathbb{Y}$ are independent random variables. Note that $P\{\mathbb{X} > n\} = P\{\text{First } n \text{ trials end in failure}\} = (1-p)^n$.

   Hence, by the law of total probability,

$$\begin{aligned}
P\{\mathbb{X} > \mathbb{Y}\} &= \sum_{n=0}^{\infty} P\{\mathbb{X} > \mathbb{Y} \mid \mathbb{Y} = n\} P\{\mathbb{Y} = n\} \\
&= \sum_{n=0}^{\infty} (1-p)^n \cdot \exp(-\lambda)\frac{\lambda^n}{n!} = \exp(-\lambda)\sum_{n=0}^{\infty} \frac{(\lambda(1-p))^n}{n!} \\
&= \exp(-\lambda)\exp(\lambda(1-p)) = \exp(-\lambda p).
\end{aligned}$$

6. (a) $f_{\mathbb{X}}(u)$ is the area of the cross-section of the pdf solid at u. Thus, obviously, $f_{\mathbb{X}}(u) = 0$ for $u \leq 0$ and $u \geq 1$. For $0 < u < 1$,

$$f_{\mathbb{X}}(u) = \int_{v=0}^{v=u} \frac{3}{2}\,dv + \int_{v=u}^{v=1} \frac{1}{2}\,dv = \frac{3u}{2} + \frac{1-u}{2} = \frac{1}{2} + u.$$

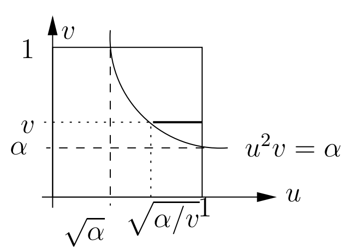
   (b) When the joint pdf has constant value over a region, we can find the probability that the random point $(\mathbb{X}, \mathbb{Y})$ lies in that region by finding the area of the region and multiplying by the pdf value. The region $\{(u,v)\colon 0 < u < 1,\ 0 < v < 1,\ u + v \leq \frac{3}{2}\}$ has area $\frac{7}{8}$ over half of which the pdf has value $\frac{1}{2}$ while over the other half, the pdf value is $\frac{3}{2}$. Hence,

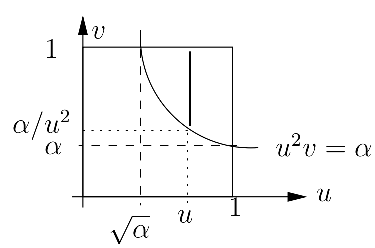
$$P\{\mathbb{X} + \mathbb{Y} \leq 3/2\} = \frac{7}{16} \times \frac{1}{2} + \frac{7}{16} \times \frac{3}{2} = \frac{7}{8}.$$

   Similarly, the region $\{(u,v)\colon 0 < u < 1,\ 0 < v < 1,\ u^2 + v^2 \geq 1\}$ has area $1 - \frac{\pi}{4}$ over half of which the pdf has value $\frac{1}{2}$ while over the other half, the pdf value is $\frac{3}{2}$. Hence,

$$P\{\mathbb{X}^2 + \mathbb{Y}^2 \geq 1\} = \frac{1}{2}\left[1 - \frac{\pi}{4}\right] \times \left(\frac{1}{2} + \frac{3}{2}\right) = 1 - \frac{\pi}{4}.$$

7. $\mathbb{Z} = \mathbb{X}^2\mathbb{Y}$ takes on values between 0 and 1.





For $0 < \alpha < 1$, $P\{\mathbb{Z} > \alpha\} = \displaystyle\int_{v=\alpha}^{1}\int_{u=\sqrt{\alpha/v}}^{1} 4uv\,du\,dv = \int_{v=\alpha}^{1} 2u^2 v\Big|_{\sqrt{\alpha/v}}^{1} dv = \int_{v=\alpha}^{1} 2(v-\alpha)\,dv$
$= (v-\alpha)^2\Big|_{\alpha}^{1} = (1-\alpha)^2$. Hence, $f_{\mathbb{Z}}(\alpha) = -\dfrac{d}{d\alpha}P\{\mathbb{Z} > \alpha\} = \begin{cases} 2(1-\alpha), & 0 < \alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$

Alternatively, $P\{\mathbb{Z} > \alpha\} = \displaystyle\int_{u=\sqrt{\alpha}}^{1}\int_{v=\alpha/u^2}^{1} 4uv\,dv\,du = \int_{u=\sqrt{\alpha}}^{1} 2u\left(1 - \frac{\alpha^2}{u^4}\right)du = u^2 + \frac{\alpha^2}{u^2}\Big|_{\sqrt{\alpha}}^{1}$
$= 1 + \alpha^2 - \alpha - \alpha = 1 - 2\alpha + \alpha^2 = (1-\alpha)^2$ etc. as before...

8. (a) For $0 \le k \le n$,

$$P\{\mathbb{X}=n,\mathbb{Y}=k\} = P\{\mathbb{Y}=k \mid \mathbb{X}=n\}\cdot P\{\mathbb{X}=n\} = \binom{n}{k}p^k(1-p)^{n-k}\cdot \exp(-\lambda)\frac{\lambda^n}{n!}.$$

(b) Since $\mathbb{Y} \le \mathbb{X}$, we use the result that $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ to write

$$\begin{aligned}
P\{\mathbb{Y}=k\} &= \sum_n P\{\mathbb{X}=n,\mathbb{Y}=k\} = \sum_{n=k}^{\infty}\binom{n}{k}p^k(1-p)^{n-k}\cdot\exp(-\lambda)\frac{\lambda^n}{n!}\\
&= \exp(-\lambda)\frac{(\lambda p)^k}{k!}\sum_{n=k}^{\infty}\frac{(\lambda(1-p))^{n-k}}{(n-k)!} = \exp(-\lambda)\frac{(\lambda p)^k}{k!}\sum_{m=0}^{\infty}\frac{(\lambda(1-p))^m}{m!}\\
&= \exp(-\lambda)\frac{(\lambda p)^k}{k!}\exp(\lambda(1-p)) = \exp(-\lambda p)\frac{(\lambda p)^k}{k!}
\end{aligned}$$

Thus, the *un*conditional pmf of $\mathbb{Y}$ is a *Poisson* pmf with parameter λp.

(c) By LOTUS, we have

$$\begin{aligned}
\mathsf{E}[\mathbb{X}\mathbb{Y}] &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} k\cdot n\cdot P\{\mathbb{X}=n,\mathbb{Y}=k\} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} k\cdot n\cdot\binom{n}{k}p^k(1-p)^{n-k}\cdot\exp(-\lambda)\frac{\lambda^n}{n!}\\
&= \sum_{n=0}^{\infty} n\cdot\exp(-\lambda)\frac{\lambda^n}{n!}\sum_{k=0}^{n}k\cdot\binom{n}{k}p^k(1-p)^{n-k}
\end{aligned}$$

But the inner sum is the one that arises in the calculation of the mean of a binomial random variable with parameters (n,p) and hence has value np. Thus we get

$$\mathsf{E}[\mathbb{X}\mathbb{Y}] = p\sum_{n=0}^{\infty} n^2\cdot\exp(-\lambda)\frac{\lambda^n}{n!} = p\cdot\mathsf{E}[\mathbb{X}^2] = p\cdot\big(\mathsf{var}(\mathbb{X}) + (\mathsf{E}[\mathbb{X}])^2\big) = p(\lambda+\lambda^2).$$

9. Obviously, $\mathbb{Z}$ takes on values only in the interval $[0,b]$ and thus $F_{\mathbb{Z}}(\alpha)=0$ for $\alpha<0$ and $F_{\mathbb{Z}}(\alpha)=1$ for $\alpha>b$. Now, for $0\le\alpha\le b$,

$$\begin{aligned}
F_{\mathbb{Z}}(\alpha) &= P\{\mathbb{Z}\le\alpha\} = P\{b-\max(\mathbb{X},\mathbb{Y})\le\alpha\} = P\{\max(\mathbb{X},\mathbb{Y})\ge(b-\alpha)\}\\
&= P\big(\{\mathbb{X}\ge(b-\alpha)\}\cup\{\mathbb{Y}\ge(b-\alpha)\}\big)\\
&= P\{\mathbb{X}\ge(b-\alpha)\} + P\{\mathbb{Y}\ge(b-\alpha)\} - P\big(\{\mathbb{X}\ge(b-\alpha)\}\cap\{\mathbb{Y}\ge(b-\alpha)\}\big)\\
&= \left(1-\frac{(b-\alpha)}{b}\right)+\left(1-\frac{(b-\alpha)}{b}\right)-\left(1-\frac{(b-\alpha)}{b}\right)^2 = \frac{2\alpha}{b}-\frac{\alpha^2}{b^2} = \frac{2b\alpha-\alpha^2}{b^2}.
\end{aligned}$$

Alternatively,
$F_{\mathbb{Z}}(\alpha) = P\{\max(\mathbb{X},\mathbb{Y})\ge(b-\alpha)\} = 1-P\{\max(\mathbb{X},\mathbb{Y})<(b-\alpha)\} = 1-P\{\mathbb{X}<b-\alpha,\mathbb{Y}<b-\alpha\}$
$= 1-\left(\dfrac{b-\alpha}{b}\right)^2 = \dfrac{2b\alpha-\alpha^2}{b^2}$ as before.