**University of Illinois** Fall 2009

# ECE 313: Final Exam

Tuesday, December 15, 2009, 8:00 a.m. — 11:00 a.m.
100 Materials Science and Engineering Building

1. **[60 points]** (3 points per answer)
Mark TRUE or FALSE for each statement below. No justification is required, but to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) $A$, $B$, and $C$ are events such that $0 < P(A) < 1$, $0 < P(B) < 1$, and $0 < P(C) < 1$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(A \cup B) = P(A^c \cup B^c) - P(A^c) - P(B^c) + 1$. |
| ☐ | ☐ | $P(AB) \geq P(A) + P(B) - 1$. |
| ☐ | ☐ | $P(A^c \mid B)P(B) + P(A^c \mid B^c)P(B^c) = P(A^c)$. |
| ☐ | ☐ | $P(A^c \mid B)P(B) + P(A \mid B)P(B) = P(B)$. |
| ☐ | ☐ | $P(A^c \mid B)P(B) + P(A \mid B^c)P(B^c) = P(A \cup B) - P(AB)$. |
| ☐ | ☐ | $P(B \mid A) = P(A \mid B)P(A)/P(B)$. |
| ☐ | ☐ | If $A$ and $B$ are *mutually exclusive* events, then they are *independent* events. |
| ☐ | ☐ | If $A$, $B$, and $C$ are *independent* events, then $P(ABC) = P(A)P(B)P(C)$. |
| ☐ | ☐ | If $P(ABC) = P(A)P(B)P(C)$, then $A$, $B$, and $C$ are *independent* events. |

(b) $\mathbb{X}$ is a continuous random variable whose probability density function $f_{\mathbb{X}}(u)$ is an *even function*, that is, $f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u)$ for *all* real numbers $u$. Let $F_{\mathbb{X}}(u)$ denote the cumulative probability distribution function (CDF) of $\mathbb{X}$, and assume that that $\mathsf{var}(\mathbb{X}) = 4$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $F_{\mathbb{X}}(u) = F_{\mathbb{X}}(-u)$ for all $u, -\infty < u < \infty$. |
| ☐ | ☐ | $\mathsf{E}[\mathbb{X}^2] = 4$. |
| ☐ | ☐ | $\mathsf{E}[\lvert\mathbb{X}\rvert] = 2$. |
| ☐ | ☐ | $P\{\mathbb{X} > u\} = F_{\mathbb{X}}(-u)$ for all $u, \ -\infty < u < \infty$. |
| ☐ | ☐ | $P\{\mathbb{X} > u\} \leq 2u^{-2}$ for all $u > 0$. |
| ☐ | ☐ | $P\{\mathbb{X}(\mathbb{X} - 1) < 2\} = P\{\mathbb{X}(\mathbb{X} + 1) < 2\}$. |

(c) $\mathbb{X}$ and $\mathbb{Y}$ are jointly continuous random variables with marginal pdfs $f_{\mathbb{X}}(u)$ and $f_{\mathbb{Y}}(v)$ respectively, and finite means and variances.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | The pdf of the sum $\mathbb{X}+\mathbb{Y}$ is $f_{\mathbb{X}+\mathbb{Y}} = f_{\mathbb{X}} \star f_{\mathbb{Y}}$ where $\star$ denotes convolution. |
| ☐ | ☐ | If $\mathbb{X}$ and $\mathbb{Y}$ are *independent* random variables, then $\mathbb{X}^2$ and $\mathbb{Y}^2$ are independent. |
| ☐ | ☐ | If $\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated* random variables, then $\mathbb{X}^2$ and $\mathbb{Y}^2$ are uncorrelated. |
| ☐ | ☐ | If $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y})$, then $\mathsf{cov}(\mathbb{X}+\mathbb{Y}, \mathbb{X}-\mathbb{Y}) = 0$. |
| ☐ | ☐ | If $\mathbb{X}$ and $\mathbb{Y}$ are *independent* random variables, then $\mathsf{var}(3\mathbb{X}+2\mathbb{Y}) = \mathsf{var}(-3\mathbb{X}+2\mathbb{Y})$. |

2. [**20 points**] Tom goes to the casino with \$5. The game consists of Tom tossing a fair coin. If the result of a toss is Heads, the casino pays Tom \$1 while if the result is Tails, Tom pays the casino \$1. Thus, after each coin toss, the amount of money that Tom has — hereinafter referred to as his *wealth* — either increases by \$1 or decreases by \$1. The (independent) coin tosses continue until *one* of the following two events occurs: **(i)** Tom's *wealth* increases to \$8 *or* **(ii)** Tom's *wealth* decreases to \$0. At this point, Tom goes home (with *wealth* \$8 or \$0 as the case may be).

   (a) [**3 points**] What is the probability that Tom goes home after three coin tosses and what is his *wealth* at this time?

   (b) [**3 points**] What is the probability that Tom goes home after five coin tosses with *wealth* \$0?

   (c) [**4 points**] What is the probability that Tom goes home after five coin tosses?

   (d) [**10 points**] *Conditioned* on the event $A = \{\text{Tom tosses the coin at least 5 times}\}$, what is the conditional pmf of $\mathbb{X}$, Tom's *wealth* after 5 tosses have occurred? Note that the increase or decrease in Tom's *wealth* due to the result of the fifth toss is included in $\mathbb{X}$.

3. [**20 points**] Dick goes to the casino with \$100. The game consists of Dick tossing a biased coin that comes up Heads with probability $\frac{1}{3}$. If the result of a toss is Heads, the casino pays Dick \$1 while if the result is Tails, Dick pays the casino \$1. Thus, after each coin toss, the amount of money that Dick has — hereinafter referred to as his *wealth* — either increases by \$1 or decreases by \$1.

   Let $\mathbb{X}$ denote Dick's *wealth* after 72 independent coin tosses have occurred. Note that the increase or decrease in Dick's *wealth* due to the result of the 72-th toss is included in $\mathbb{X}$.

   (a) [**8 points**] Find the mean and variance of $\mathbb{X}$.

   (b) [**12 points**] *Estimate* the probability that after 72 coin tosses, Dick's *wealth* is *at least* \$64. Your answer need not be exact, but you must explain how you arrived at your answer.

4. [**20 points**] Harry is looking for a soulmate on the social network Facespace, which suggests a randomly chosen Facespace member (of the apposite sex) to Harry. Assume that all suitable Facespace members are equally likely to be born under one of the 12 signs of the Zodiac. Harry accepts Facespace's suggestion if the person suggested has the *same sign as Harry*, or the *sign that precedes Harry's sign*, or the *sign that follows Harry's sign* in the traditional arrangement of the signs of the Zodiac in a circular pattern. Otherwise, Harry rejects the proposed candidate and Facespace suggests another member, chosen at random from the same set and independent of the previous choice. (Note: this is sampling with replacement). The suggestions continue till Harry finds someone with whom he is astrologically compatible.

   (a) [**6 points**] What is the average number of suggestions that are *rejected* by Harry before he accepts a suggestion from Facespace?

   (b) [**14 points**] Dissatisfied with the candidates proposed by Facespace, Harry turns to Mybook which secretly classifies one-thirds of its members as $H$-type ("Hot") and two-thirds as $N$-type ("Not"). Mybook presents a member selected at random from its subscribers to Harry who accepts a $H$-type candidate with probability $\frac{3}{4}$ and an $N$-type candidate with probability $\frac{3}{40}$.

   - What is the probability that Harry accepts the proposed candidate?
   - What is the probability that an accepted candidate is a $H$-type?

5. **[20 points]** Let $\mathbb{X}$ and $\mathbb{Y}$ be *independent discrete* random variables where $\mathbb{X}$ is a *geometric* random variable with parameter $p$ and $\mathbb{Y}$ is a *Poisson* random variable with parameter $\lambda$.

   Find $P\{\mathbb{X} > \mathbb{Y}\}$.

   Hint: Find the desired probability conditioned on $\{\mathbb{Y} = n\}$ and then use the law of total probability,

6. **[20 points]** The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf given by

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} \frac{1}{2}, & 0 < u < v < 1, \\ \frac{3}{2}, & 0 < v < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) **[8 points]** Find the marginal pdf of $\mathbb{X}$. In order to receive full credit, you must specify the value of $f_{\mathbb{X}}(u)$ for all $u, -\infty < u < \infty$.

   (b) **[12 points]** Find $P\{\mathbb{X} + \mathbb{Y} \leq 3/2\}$ and $P\{\mathbb{X}^2 + \mathbb{Y}^2 \geq 1\}$.

7. **[20 points]** The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf given by

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} 4uv, & 0 < u < 1, \quad 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   Find the pdf of $\mathbb{Z} = \mathbb{X}^2\mathbb{Y}$.

   In order to receive full credit, you must specify the value of $f_{\mathbb{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.

8. **[25 points]** Let $\mathbb{X}$ be a Poisson random variable with parameter $\lambda$. Given that $\mathbb{X} = n$, the conditional pmf of the random variable $\mathbb{Y}$ is a binomial pmf with parameters $(n, p)$.

   (a) **[4 points]** Find $P\{\mathbb{X} = n, \mathbb{Y} = k\}$ for non-negative integers $k, n$ with $k \leq n$.

   (b) **[8 points]** Find $P\{\mathbb{Y} = k\}$ for all non-negative integers $k$.

   (c) **[13 points]** Find $\mathsf{E}[\mathbb{X}\mathbb{Y}]$. Your answer should depend only on $\lambda$ and $p$.

9. **[20 points]** $\mathbb{X}$ and $\mathbb{Y}$ are independent random variables and each is uniformly distributed over the interval $[0, b]$. Find the cdf $F_{\mathbb{Z}}(\alpha)$ of the random variable $\mathbb{Z} = b - \max(\mathbb{X}, \mathbb{Y})$. To obtain full credit, you must specify the value of $F_{\mathbb{Z}}(\alpha)$ for all $\alpha$, $-\infty < \alpha < \infty$.