1. [15 points] Let $A$, $B$, and $C$ denote three events defined on a sample space $\Omega$, and suppose that $P(A) = 0.6$, $P(B) = P(C) = 0.3$, and $P(B^c \cap C) = P(A \cap B^c \cap C^c) = 0.2$.

**Solution:** The Karnaugh maps shown below are very useful in visualizing the problem.

(a) [5 points] Find $P(B \cap C)$.

**Solution:**

\[
P(B \cap C) = P(C) - P(B^c \cap C) = 0.3 - 0.2 = 0.1.
\]

Note that $P(B \cup C) = P(B) + P(B^c \cap C) = 0.5$.

(b) [5 points] Find $P(B \cap C^c)$.

**Solution:**

\[
P(B \cap C^c) = P(B) - P(B \cap C) = 0.3 - 0.1 = 0.2.
\]

(c) [5 points] Find $P((A \cup B \cup C)^c)$.

**Solution:**

\[
P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - [P(B \cup C) + P(A \cap B^c \cap C^c)]
\]

\[
= 1 - [0.5 + 0.2] = 0.3.
\]

Note that $P((A \cup B \cup C)^c) = P(A^c \cap B^c \cap C^c)$, and also that $P(B^c \cap C^c) = 1 - P(B \cup C) = 0.5$.

2. [10 points] $A$ and $B$ are events defined on a sample space $\Omega$. Assume $P(A), P(B) > 0$. Mark each of the two statements below as TRUE or FALSE. No justification is needed.

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If $P(A \mid B) = P(B \mid A)$, then $P(A) = P(B)$.

**Solution:** The statement “If $P(A \mid B) = P(B \mid A)$, then $P(A) = P(B)$” seems to be true at first glance, but is in fact FALSE in general. (The statement is true if we are also told that $P(A \cap B) > 0$.)

The statement “$P(A \mid B)P(B) + P(A^c \mid B)P(B) = P(B)$” is TRUE. The left side is equal to $P(A \cap B) + P(A^c \cap B)$ which is the same as $P(B)$ by the third axiom.
3. [30 points] Especially in this problem, you must provide sufficient explanation to justify your numerical answers.

A fair coin is tossed repeatedly until a Head occurs. $N$ denotes the number of tosses.

Solution: Clearly, $N$ is a geometric random variable with parameter $p = \frac{1}{2}$.

(a) [5 points] What is the expected value of $N$?

Solution: Since $N$ is a geometric random variable with parameter $p = \frac{1}{2}$, $E[N] = \frac{1}{p} = 2$ using a formula that you might have on your sheet of notes.

Alternatively, $P\{N = k\} = 2^{-k}$ for $k \geq 1$, and hence

$$E[N] = 1 \cdot 2^{-1} + 2 \cdot 2^{-2} + 3 \cdot 2^{-3} + \cdots = \frac{1}{2} \left[ 1 + 2 \cdot 2^{-1} + 3 \cdot 2^{-2} + \cdots \right] = \frac{1}{2} \left( \frac{1}{1 - \frac{1}{2}} \right)^2 = 2$$

where the series sum was found on Problem Set 0, and its importance in ECE 313 noted.

(b) [5 points] Find the numerical value of $P\{N > 5\}$.

Solution: Since $N$ is a geometric random variable with parameter $p = \frac{1}{2}$, $P\{N > 5\} = (1 - p)^5 = \frac{1}{32}$ using a formula that you might have on your sheet of notes.

Alternatively, $N > 5$ if and only if the first five tosses resulted in Tails, and this has probability $\left( \frac{1}{2} \right)^5 = \frac{1}{32}$.

Doing it an even harder way,

$$P\{N > 5\} = 2^{-6} + 2^{-7} + 2^{-8} + \cdots = 2^{-6} \left[ 1 + 2^{-1} + 2^{-2} + \cdots \right] = 2^{-6} \cdot \frac{1}{1 - \frac{1}{2}} = 2^{-5}$$

Yowza!

(c) [10 points] Given that the event $P\{N > 5\}$ occurred, what is the expected value of $N$?

Solution: $N$ is the waiting time for a Head to occur, and $E[N]$ is the average waiting time for a Head to occur. Given that the event $P\{N > 5\}$ occurred, that is, the first five tosses resulted in Tails, the waiting time for a Head (beginning with the 6th toss) is still the same by the memoryless property of the geometric distribution. Hence, $E[N \mid N > 5] = 5 + 2 = 7$ using the result from part (a).

Alternatively, conditioned on Tails on the first five tosses,

$P\{N = 6\} = 2^{-1}, P\{N = 7\} = 2^{-2}, P\{N = 8\} = 2^{-3}$ and so on. Hence, the average value of $N$ conditioned on Tails on the first five tosses is $6 \cdot 2^{-1} + 7 \cdot 2^{-2} + 8 \cdot 2^{-3} + \cdots = (5 + 1) \cdot 2^{-1} + (5 + 2) \cdot 2^{-2} + (5 + 3) \cdot 2^{-3} + \cdots = 5 \cdot \left[ \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots \right] + 1 \cdot \frac{1}{2} + 2 \cdot 2^{-2} + 3 \cdot 2^{-3} + \cdots = 5 + 2 = 7$ using the sum evaluated in part (a).

(d) [10 points] Find the numerical value of $E[\cos(\pi N)]$.

Solution: $\cos(\pi N) = \begin{cases} -1 & \text{if } N \text{ is odd}, \\ +1 & \text{if } N \text{ is even}, \end{cases}$

$$E[\cos(\pi N)] = \frac{1}{1 - (-\frac{1}{2})} - 1 = -\frac{1}{3} = P(\text{Wilma wins}) - P(\text{Fred wins})$$

when they play with a fair coin.

4. [20 points] A fair coin is tossed 10 times.

Calculate the probability that the first 5 tosses are all Tails given that a total of 8 Tails occurred on the 10 tosses.

Solution: Let $A$ denote the event that the first 5 tosses result in Tails, and $B$ the event that 8 Tails occurred on the 10 tosses. We are asked to find $P(A \mid B) = P(A \cap B) / P(B)$. Now, the number of Tails on 10 tosses is a binomial random variable $N$ with parameters $(10, \frac{1}{2})$, and thus, we have

$$P(B) = P\{N = 8\} = \binom{10}{8} \left( \frac{1}{2} \right)^8 \left( 1 - \frac{1}{2} \right)^{10-8} = \binom{10}{2} \left( \frac{1}{2} \right)^{10} = \frac{10 \times 9}{1 \times 2} \times 2^{-10}.$$
On the other hand, \( A \cap B = A \cap C \) where \( C \) is the event that 3 Tails occurred on the last 5 tosses. Since what occurred on the last five tosses is independent of what occurred on the first five tosses, we have that

\[
P(A \cap B) = P(A \cap C) = 2^{-5} \left( \frac{5}{3} \right) \left( \frac{1}{2} \right)^3 \left( \frac{1}{2} \right)^{5-3} = \left( \frac{5}{2} \right) \left( \frac{1}{2} \right)^{10} = \frac{5 \times 4}{1 \times 2} \times 2^{-10}.
\]

Hence, \( P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{5 \times 4}{10 \times 9} = \frac{2}{9}. \)

5. [25 points] Dilbert has 3 coins in his pocket, 2 of which are fair coins while the third is a biased coin with \( P(H) = p \neq \frac{1}{2} \). The probability that a coin chosen at random from his pocket will land Tails is \( \frac{7}{12} \).

(a) [10 points] What is the value of \( p \)?

Solution: \[ P(T) = P(T \mid \text{fair coin})P(\text{fair coin}) + P(T \mid \text{biased coin})P(\text{biased coin}) = \frac{1}{2} \cdot \frac{2}{3} + (1-p) \cdot \frac{1}{3} = \frac{7}{12} \Rightarrow p = \frac{1}{4}. \]

(b) [15 points] Dilbert picks two coins at random from his pocket, tosses each coin once, and observes a Head and a Tail. What is the conditional probability that both coins are fair?

Solution:

\[
P(\text{one Head, one Tail} \mid \text{two fair coins}) = P(\{HT, TH\}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2},
\]

\[
P(\text{one Head, one Tail} \mid \text{one fair, one biased}) = P(\{\text{fair = H, biased = T}\}) + P(\{\text{biased = H, fair = T}\}) = \frac{1}{2} \cdot (1-p) + p \cdot \frac{1}{2} = \frac{1}{2}
\]

regardless of the value of \( p \). Therefore, the theorem of total probability gives

\[
P(\text{one Head, one Tail}) = P(H, T \mid \text{two fair coins})P(\text{two fair coins}) + P(H, T \mid \text{one fair, one biased})P(\text{one fair, one biased})
= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2},
\]

and the conditional probability of both coins being fair given that one Head and one Tail was observed is, by Bayes’ formula,

\[
P(\text{both coins fair} \mid \text{one Head, one Tail}) = \frac{P(H, T \mid \text{two fair coins})P(\text{two fair coins})}{P(\text{one Head, one Tail})} = \frac{1/6}{1/2} = \frac{1}{3}
\]

the same as the unconditional probability! What does this tell you about the events \{both coins fair\} and \{one Head, one Tail\}?