1. [15 points] Messages can be transmitted between Champaign-Urbana and Springfield either
directly or via Decatur on communication links whose capacities are indicated on the diagram
shown below. Assume that the three links fail independently with equal probability \( \frac{1}{2} \). Let
\( X \) denote the communication capacity between Champaign-Urbana and Springfield.

   ![Diagram of communication links]

(a) [5 points] What values can \( X \) take on?
   Solution: It is easy to get that \( X \) can take on values 0, 40, 200, and 240.

(b) [10 points] Find the expected value of \( X \).
   Solution: \( X \) has value 240 when all links are working, which occurs with probability \( \frac{1}{8} \).
   \( X \) has value 40 when the direct link between CU and Springfield has failed while the
   other two links are working, which also occurs with probability \( \frac{1}{8} \).
   \( X \) has value 200 when the direct link between CU and Springfield is working while at
   least one of the other two links has failed, which occurs with probability \( \frac{3}{8} \). Hence,
   \[
   E[X] = 240 \times \frac{1}{8} + 200 \times \frac{3}{8} + 40 \times \frac{1}{8} = 240 \times \frac{1}{8} + 200 \times \frac{3}{8} + 40 \times \frac{1}{8} = \frac{880}{8} = 110.
   \]

2. [10 points] Let \( X \) denote a Gaussian random variable with mean 2 and variance 25.
   Use the table of values of the unit Gaussian CDF \( \Phi(\cdot) \) on the last page of this exam booklet
to find the numerical value of \( P\{|X-4| > 3\} \).
   Solution: \( P\{|X-4| > 3\} = 1 - P\{|X-4| \leq 3\} = 1 - P\{1 \leq X \leq 7\} = 1 - \left[ \Phi\left(\frac{7-2}{5}\right) - \Phi\left(\frac{1-2}{5}\right) \right] = 1 - [\Phi(1) - \Phi(-0.2)] = 2 - \Phi(1) - \Phi(0.2) = 2 - 0.8413 - 0.5793 = 0.5794.

3. [15 points] Let \( X \) denote a continuous random variable with zero mean and variance 1, that
is, \( E[X] = 0 \) and \( \text{var}(X) = 1 \).
   (a) [10 points] Find the mean and variance of \( Y = 2X + 3 \).
      Solution: \( E[Y] = E[2X + 3] = 2E[X] + 3 = 3. \text{var}(Y) = \text{var}(2X + 3) = 2^2\text{var}(X) = 4. \)
   (b) [5 points] Let \( Z = |X| \).
      Mark TRUE or FALSE by checking one box below. You need not justify your answer.
      \[
      \begin{array}{ll}
      \text{TRUE} & \text{FALSE} \\
      \hline
      \square & \square \quad \text{The variance of } Z \text{ is greater than 1, that is } \text{var}(Z) > 1.
      \end{array}
      \]
4. [10 points] Let $X$ denote an exponential random variable with mean 2.
   
   (a) [5 points] Find $P\{X > 1\}$.
   
   **Solution:** The pdf of $X$ is $\frac{1}{2} \exp(-u/2)$ for $u > 0$.
   
   Hence, $P\{X > 1\} = \int_1^{\infty} \frac{1}{2} \exp(-u/2) \, du = - \exp(-u/2) \bigg|_1^{\infty} = \exp(-1/2)$.
   
   This result could also be obtained from one that you might have on your sheet of notes: $P\{X > T\} = R_X(T) = \exp(-T/\mu)$ for an exponential random variable with mean $\mu$.
   
   (b) [5 points] Calculate $P\{X > 3 \mid X > 2\}$.
   
   **Solution:** We can use the memoryless property of exponential random variables to write $P\{X > 3 \mid X > 2\} = P\{X > 3 - 2\} = P\{X > 1\} = \exp(-1/2)$ without doing any calculations at all! More painstakingly,
   
   $$P\{X > 3 \mid X > 2\} = \frac{P\{(X > 3) \cap (X > 2)\}}{P\{X > 2\}} = \frac{P\{X > 3\}}{P\{X > 2\}} = \frac{\exp(-3/2)}{\exp(-2/2)} = \exp(-1/2).$$
   
5. [25 points] $X$ denotes a uniform random variable with mean 1 and variance 3.
   
   (a) [10 points] Find $P\{X < 0\}$.
   
   **Solution:** A random variable uniformly distributed on $[a, b]$ has mean $\frac{a + b}{2}$ and variance $\frac{(b - a)^2}{12}$. Hence, we have that $b - a = 6$, and $b + a = 2$, giving $a = -2, b = 4$. The pdf is thus as shown in the left-hand figure below.
   
   ![Uniform Distribution](image)
   
   By inspection, $P\{X < 0\} = \frac{1}{3}$
   
   (b) [15 points] Find the pdf of $Y = |X|$. In order to receive full credit, you must specify the value of $f_Y(v)$ for all $v, -\infty < v < \infty$.
   
   **Solution:** $Y = |X|$ takes on values in $[0, 4]$, and hence $F_Y(v) = 0$ for $v < 0$ and $F_Y(v) = 1$ for $v > 4$.
   
   For any $v, 0 \leq v \leq 2$, $F_Y(v) = P\{Y \leq v\} = P\{-v \leq X \leq v\} = v/3$.
   
   For any $v, 2 \leq v \leq 4$, $F_Y(v) = P\{Y \leq v\} = P\{X \leq v\} = (v + 2)/6$.
   
   Hence, $f_Y(v) = \frac{d}{du} F_Y(v) =$ \begin{align*}
   \frac{1}{3}, & \quad 0 \leq v \leq 2, \\
   \frac{1}{6}, & \quad 2 < v \leq 4, \\
   0, & \quad \text{elsewhere}.
\end{align*}
   
   It is easy to verify that this is a valid pdf, and thus we have not made any obvious errors.
6. [25 points] Consider the following binary hypothesis testing problem.

If hypothesis $H_0$ is true, the continuous random variable $X$ is uniformly distributed on $(-2, 2)$, while if hypothesis $H_1$ is true, the pdf of $X$ is $f_1(u) = \begin{cases} \frac{|u|}{4}, & |u| < 2, \\ 0, & \text{otherwise}. \end{cases}$

**Solution:** The easiest way to solve this problem is to first sketch the two pdfs as shown in the left-hand figure below.

(a) [5 points] Find the decision region $\Gamma_0$ for the maximum-likelihood decision rule. Remember that $\Gamma_0$ is the set of all real numbers such that if $X \in \Gamma_0$, the decision is that $H_0$ is the true hypothesis.

**Solution:** By inspection of the figure, we see that $f_0(u) > f_1(u)$ for $|u| < 1$, and hence $\Gamma_0 = \{u: |u| < 1\} = (-1, 1)$. Note that either or both end-points $\pm 1$ can be included in $\Gamma_0$ if desired.

The graphically-challenged can proceed as follows.

For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = |u|$.

When $X = u$ is the observation, the maximum-likelihood decision rule decides in favor of $H_0$ if $\Lambda(u) < 1$, that is, $|X| < 1$. Hence $\Gamma_0 = \{u: |u| < 1\} = (-1, 1)$.

(b) [10 points] Find the probability of false alarm $P_{FA}$ and the probability of missed detection $P_{MD}$ for the maximum-likelihood decision rule.

**Solution:** By inspection of the figure, we get that $P_{FA} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{MD} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

Alternatively, $P_{FA} = \int_{\Gamma_0} f_0(u) \, du = 2 \int_{-1}^{1} \frac{1}{4} \, du = \frac{1}{2}$.

$P_{MD} = \int_{\Gamma_0} f_1(u) \, du = \int_{-1}^{1} \frac{1}{4} |u| \, du = \frac{1}{2} \int_{0}^{1} u \, du = \frac{1}{4}$.

(c) [10 points] Suppose that the hypotheses have a priori probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. Find the decision region $\Gamma_0$ for the maximum a posteriori probability (MAP) decision rule (also known as the minimum-error-probability or Bayesian decision rule).

**Solution:** Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of $H_0$ if $|X| < 0.5$. Thus, $\Gamma_0 = \{u: |u| < \frac{1}{2}\} = (-\frac{1}{2}, \frac{1}{2})$ for the MAP decision rule.

Alternatively, the MAP rule decides that $H_0$ is the true hypothesis when the likelihood ratio $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = |u| < \frac{\pi_0}{\pi_1} = \frac{1}{2}$. Hence the MAP decision rule has decision region $\Gamma_0 = \{u: |u| < \frac{1}{2}\} = (-\frac{1}{2}, \frac{1}{2})$. 

3