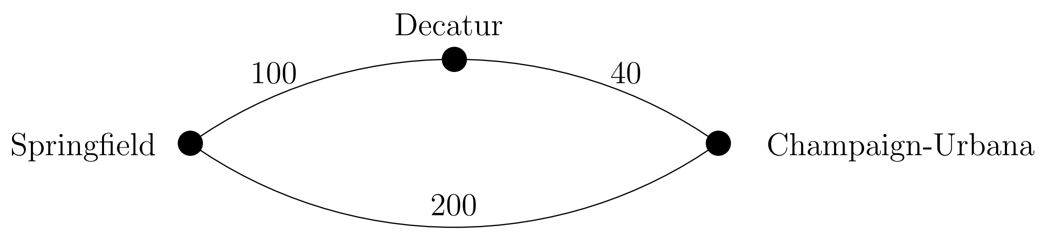
University of Illinois Fall 2009

ECE 313: Hour Exam II

Monday November 16, 2009
7:00 p.m. — 8:00 p.m.
100 Materials Science and Engineering Building

1. **[15 points]** Messages can be transmitted between Champaign-Urbana and Springfield either directly or via Decatur on communication links whose capacities are indicated on the diagram shown below. Assume that the three links fail independently with equal probability $\frac{1}{2}$. Let $\mathbb{X}$ denote the communication capacity between Champaign-Urbana and Springfield.



   (a) **[5 points]** What values can $\mathbb{X}$ take on?
   Solution: It is easy to get that $\mathbb{X}$ can take on values 0, 40, 200, and 240.

   (b) **[10 points]** Find the expected value of $\mathbb{X}$.
   Solution: $\mathbb{X}$ has value 240 when all links are working, which occurs with probability $\frac{1}{8}$.
   $\mathbb{X}$ has value 40 when the direct link between CU and Springfield has failed while the other two links are working, which also occurs with probability $\frac{1}{8}$.
   $\mathbb{X}$ has value 200 when the direct link between CU and Springfield is working while *at least one* of the other two links has failed, which occurs with probability $\frac{3}{8}$. Hence,

$$\mathsf{E}[\mathbb{X}] = 240 \times \frac{1}{8} + 200 \times \frac{3}{8} + 40 \times \frac{1}{8} = \frac{240 + 3 \times 200 + 40}{8} = \frac{880}{8} = 110.$$

2. **[10 points]** Let $\mathbb{X}$ denote a *Gaussian* random variable with mean 2 and variance 25.

   Use the table of values of the unit Gaussian CDF $\Phi(\cdot)$ on the last page of this exam booklet to find the numerical value of $P\{|\mathbb{X} - 4| > 3\}$.

   Solution: $P\{|\mathbb{X}-4| > 3\} = 1-P\{|\mathbb{X}-4| \le 3\} = 1-P\{1 \le \mathbb{X} \le 7\} = 1-\left[\Phi\left(\frac{7-2}{5}\right) - \Phi\left(\frac{1-2}{5}\right)\right]$
   $= 1 - [\Phi(1) - \Phi(-0.2)] = 2 - \Phi(1) - \Phi(0.2) = 2 - 0.8413 - 0.5793 = 0.5794.$

3. **[15 points]** Let $\mathbb{X}$ denote a continuous random variable with zero mean and variance 1, that is, $\mathsf{E}[\mathbb{X}] = 0$ and $\mathsf{var}(\mathbb{X}) = 1$.

   (a) **[10 points]** Find the mean and variance of $\mathbb{Y} = 2\mathbb{X} + 3$.
   Solution: $\mathsf{E}[\mathbb{Y}] = \mathsf{E}[2\mathbb{X} + 3] = 2\mathsf{E}[\mathbb{X}] + 3 = 3$. $\mathsf{var}(\mathbb{Y}) = \mathsf{var}(2\mathbb{X} + 3) = 2^2\mathsf{var}(\mathbb{X}) = 4$.

   (b) **[5 points]** Let $\mathbb{Z} = |\mathbb{X}|$.
   Mark TRUE or FALSE by checking one box below. You need not justify your answer.

TRUE	FALSE	
☐	☐	The variance of $\mathbb{Z}$ is greater than 1, that is $\mathsf{var}(\mathbb{Z}) > 1$.

Solution: The statement is false. Intuitively, since $\mathbb{Z}$ has "smaller" range than $\mathbb{X}$, its variance must be smaller than the variance of $\mathbb{X}$: the probability masses are less spread out and so the moment of inertia must be smaller. A more long-winded explanation is that $\mathsf{E}[\mathbb{Z}] > 0$ (why?) and since $\mathsf{var}(\mathbb{X}) = \mathsf{E}[\mathbb{X}^2]$ (why?), we have that

$$\mathsf{var}(\mathbb{Z}) = \mathsf{E}[\mathbb{Z}^2] - (\mathsf{E}[\mathbb{Z}])^2 = \mathsf{E}[\mathbb{X}^2] - (\mathsf{E}[\mathbb{Z}])^2 = \mathsf{var}(\mathbb{X}) - (\mathsf{E}[\mathbb{Z}])^2 < \mathsf{var}(\mathbb{X}).$$

4. **[10 points]** Let $\mathbb{X}$ denote an *exponential* random variable with mean 2.

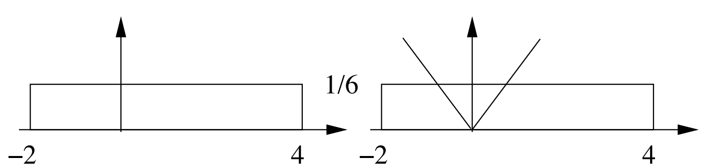
   (a) **[5 points]** Find $P\{\mathbb{X} > 1\}$.
   Solution: The pdf of $\mathbb{X}$ is $\frac{1}{2}\exp(-u/2)$ for $u > 0$.
   Hence, $P\{\mathbb{X} > 1\} = \int_1^\infty \frac{1}{2}\exp(-u/2)\,du = -\exp(-u/2)\Big|_1^\infty = \exp(-1/2)$.
   This result could also be obtained from one that you might have on your sheet of notes: $P\{\mathbb{X} > T\} = R_{\mathbb{X}}(T) = \exp(-T/\mu)$ for an exponential random variable with mean μ.

   (b) **[5 points]** Calculate $P\{\mathbb{X} > 3 \mid \mathbb{X} > 2\}$.
   Solution: We can use the *memoryless* property of exponential random variables to write $P\{\mathbb{X} > 3 \mid \mathbb{X} > 2\} = P\{\mathbb{X} > 3-2\} = P\{\mathbb{X} > 1\} = \exp(-1/2)$ without doing any calculations at all! More painstakingly,

$$P\{\mathbb{X} > 3 \mid \mathbb{X} > 2\} = \frac{P[\{\mathbb{X} > 3\} \cap \{\mathbb{X} > 2\}]}{P\{\mathbb{X} > 2\}} = \frac{P\{\mathbb{X} > 3\}}{P\{\mathbb{X} > 2\}} = \frac{\exp(-3/2)}{\exp(-2/2)} = \exp(-1/2).$$

5. **[25 points]** $\mathbb{X}$ denotes a *uniform* random variable with mean 1 and variance 3.

   (a) **[10 points]** Find $P\{\mathbb{X} < 0\}$.
   Solution: A random variable uniformly distributed on $[a, b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b - a = 6$, and $b + a = 2$, giving $a = -2, b = 4$. The pdf is thus as shown in the left-hand figure below.



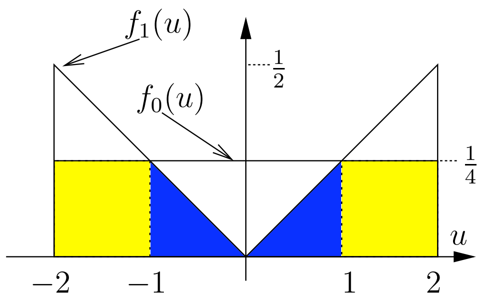
   By inspection, $P\{\mathbb{X} < 0\} = \frac{1}{3}$

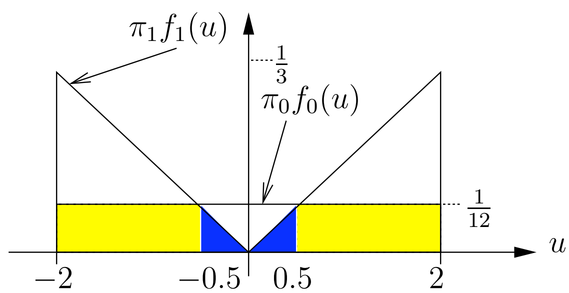
   (b) **[15 points]** Find the pdf of $\mathbb{Y} = |\mathbb{X}|$. In order to receive full credit, you must specify the value of $f_{\mathbb{Y}}(v)$ for all $v, -\infty < v < \infty$.
   Solution: $\mathbb{Y} = |\mathbb{X}|$ takes on values in $[0, 4]$, and hence $F_{\mathbb{Y}}(v) = 0$ for $v < 0$ and $F_{\mathbb{Y}}(v) = 1$ for $v > 4$.
   For any $v, 0 \le v \le 2, F_{\mathbb{Y}}(v) = P\{\mathbb{Y} \le v\} = P\{[-v \le \mathbb{X} \le v\} = v/3$.
   For any $v, 2 \le v \le 4, F_{\mathbb{Y}}(v) = P\{\mathbb{Y} \le v\} = P\{\mathbb{X} \le v\} = (v+2)/6$.
   Hence, $f_{\mathbb{Y}}(v) = \dfrac{d}{du}F_{\mathbb{Y}}(v) = \begin{cases} \frac{1}{3}, & 0 \le v \le 2, \\ \frac{1}{6}, & 2 < v \le 4, \\ 0, & \text{elsewhere.} \end{cases}$
   It is easy to verify that this is a valid pdf, and thus we have not made any obvious errors.

6. **[25 points]** Consider the following binary hypothesis testing problem.

If hypothesis H_0 is true, the continuous random variable $\mathbb{X}$ is uniformly distributed on $(-2, 2)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is $f_1(u) = \begin{cases} \frac{|u|}{4}, & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

Solution: The easiest way to solve this problem is to first sketch the two pdfs as shown in the left-hand figure below.





(a) **[5 points]** Find the decision region Γ_0 for the *maximum-likelihood* decision rule. Remember that Γ_0 is the set of all real numbers such that if $\mathbb{X} \in \Gamma_0$, the decision is that H_0 is the true hypothesis.

Solution: By inspection of the figure, we see that $f_0(u) > f_1(u)$ for $|u| < 1$, and hence $\Gamma_0 = \{u \colon |u| < 1\} = (-1, 1)$. Note that either or both end-points ± 1 can be included in Γ_0 if desired.

The graphically-challenged can proceed as follows.

For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = |u|$.

When $\mathbb{X} = u$ is the observation, the *m*aximum-likelihood decision rule decides in favor of H_0 if $\Lambda(u) < 1$, that is, $|\mathbb{X}| < 1$. Hence $\Gamma_0 = \{u \colon |u| < 1\} = (-1, 1)$.

(b) **[10 points]** Find the probability of false alarm P_{FA} and the probability of missed detection P_{MD} for the maximum-likelihood decision rule.

Solution: By inspection of the figure, we get that $P_{\text{FA}} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

Alternatively, $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = 2\int_1^2 \frac{1}{4}\,du = \frac{1}{2}$.

$P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\,du = \int_{-1}^{1} \frac{1}{4}|u|\,du = \frac{1}{2}\int_0^1 u\,du = \frac{1}{4}$.

(c) **[10 points]** Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. Find the decision region Γ_0 for the maximum *a posteriori* probability (MAP) decision rule (also known as the minimum-error-probability or Bayesian decision rule).

Solution: Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_0 if $|\mathbb{X}| < 0.5$. Thus, $\Gamma_0 = \left\{u \colon |u| < \frac{1}{2}\right\} = \left(-\frac{1}{2}, \frac{1}{2}\right)$ for the MAP decision rule.

Alternatively, the MAP rule decides that H_0 is the true hypothesis when the likelihood ratio $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = |u| < \dfrac{\pi_0}{\pi_1} = \dfrac{1}{2}$. Hence the *M*AP decision rule has decision region $\Gamma_0 = \left\{u \colon |u| < \frac{1}{2}\right\} = \left(-\frac{1}{2}, \frac{1}{2}\right)$.