University of Illinois Fall 2008

ECE 313: Problem Set 13

Covariance and Correlation, Minimum-Mean-Square-Error Estimation, Jointly Gaussian Random Variables

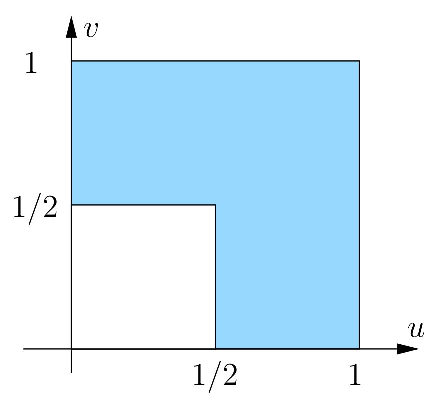
This Problem Set contains five problems

Due:	Wednesday December 10, 4 p.m..
Reading:	Ross, Chapter 7, Sections 1-5, Chapter 8, Sections 1-4
Noncredit exercises:	Ross Chapter 7: Problems 1, 16, 26, 30 33, 34, 38;
	Theoretical Exercises: 1, 2, 17, 22, 23, 40
	Chapter 8: problems 1-9, 15.

1. Let $\mathsf{E}[\mathcal{X}] = 1, \mathsf{E}[\mathcal{Y}] = 4, \mathsf{var}(\mathcal{X}) = 4, \mathsf{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X},\mathcal{Y}} = 0.1$.
   (a) If $\mathcal{Z} = 2(\mathcal{X}+\mathcal{Y})(\mathcal{X}-\mathcal{Y})$, what is $\mathsf{E}[\mathcal{Z}]$?
   (b) If $\mathcal{T} = 2\mathcal{X}+\mathcal{Y}$ and $\mathcal{U} = 2\mathcal{X}-\mathcal{Y}$, what is $\mathsf{cov}(\mathcal{T},\mathcal{U})$?
   (c) Find the mean and variance of $\mathcal{W} = 3\mathcal{X}+\mathcal{Y}+2$.
   (d) If $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, and $\mathcal{W}$ is as defined in part (c), what is $P\{\mathcal{W} > 0\}$?

2. This problem has three independent parts. Do not apply the numbers from one part to the others.
   (a) If $\mathsf{var}(\mathcal{X}+\mathcal{Y}) = 36$ and $\mathsf{var}(\mathcal{X}-\mathcal{Y}) = 64$, what is $\mathsf{cov}(\mathcal{X},\mathcal{Y})$? If you are also told that $\mathsf{var}(\mathcal{X}) = 3 \cdot \mathsf{var}(\mathcal{Y})$, what is $\rho_{\mathcal{X},\mathcal{Y}}$?
   (b) If $\mathsf{var}(\mathcal{X}+\mathcal{Y}) = \mathsf{var}(\mathcal{X}-\mathcal{Y})$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?
   (c) If $\mathsf{var}(\mathcal{X}) = \mathsf{var}(Y)$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?

3. The random point $(\mathcal{X},\mathcal{Y})$ is uniformly distributed on the shaded region shown.



   (a) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Find $\mathsf{E}[\mathcal{X}]$ and $\mathsf{var}(\mathcal{X})$.
   (b) *Write down* the marginal pdf $f_{\mathcal{Y}}(v)$ of the random variable $\mathcal{Y}$, and its mean and variance, from your answer to part (a).
   (c) Find $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$, the conditional pdf of $\mathcal{Y}$ given that $\mathcal{X} = \alpha$, where $0 < \alpha < 1/2$. *Write down* the conditional mean and conditional variance of $\mathcal{Y}$ given $\mathcal{X} = \alpha$.
   Find $f_{\mathcal{Y}|\mathcal{X}}(u|\alpha)$, the conditional pdf of $\mathcal{Y}$ given that $\mathcal{X} = \alpha$, where $1/2 < \alpha < 1$. *Write down* the conditional mean and conditional variance of $\mathcal{Y}$ given $\mathcal{X} = \alpha$.

(d) Now, apply the theorem of total probability to compute $f_{\mathcal{Y}}(v)$, the *unconditional* pdf of $\mathcal{Y}$ from $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$. Do you get the same answer as in part (b)? Why not?

(e) *Given* that the value of $\mathcal{X}$ is u, the (conditional) minimum-mean-square-error estimate of $\mathcal{Y}$ is $\mathsf{E}[\mathcal{Y}|\mathcal{X}=u]$, the conditional mean of $\mathcal{Y}$, and the (conditional minimum) mean-square error achieved is $\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)$.

Use your answers to part (c) to sketch graphs of $\mathsf{E}[\mathcal{Y}|\mathcal{X}=u]$ and $\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)$ as functions of u for $0<u<1$.

(f) Since $\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)$ depends on the value of $\mathcal{X}$, it is a *function* of $\mathcal{X}$. Use LOTUS to compute $\mathsf{E}[\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)]$, the expected value of this function. This is the (unconditional) mean-square error that we achieve when we estimate $\mathcal{Y}$ as $\mathsf{E}[\mathcal{Y}|\mathcal{X}=u]$, and *no* other estimate can have smaller mean-square error than this.

(g) It can be easily shown that $\mathsf{cov}(\mathcal{X},\mathcal{Y}) = -\dfrac{1}{36}$ and $\rho_{\mathcal{X},\mathcal{Y}} = -\dfrac{4}{11}$. Now, the minimum-mean-square-error *linear* estimate of $\mathcal{Y}$ given that the value of $\mathcal{X}$ is u is

$$\hat{\mathcal{Y}} = \mathsf{E}[\mathcal{X}] + \rho_{\mathcal{X},\mathcal{Y}}\sqrt{\mathsf{var}(\mathcal{Y})/\mathsf{var}(\mathcal{X})}\big(u-\mathsf{E}[\mathcal{X}]\big) = -\frac{4}{11}u+\frac{35}{44},$$

and the (unconditional) mean-square error of this linear estimate is $\mathsf{var}(\mathcal{Y})(1-\rho^2_{\mathcal{X},\mathcal{Y}})$.

Sketch $\hat{\mathcal{Y}}$ as a function of u on the same graph that you used in part (e) and compare $\hat{\mathcal{Y}}$ to the nonlinear (optimum) estimate $\mathsf{E}[\mathcal{Y}|\mathcal{X}=u]$. Which is larger, $\hat{\mathcal{Y}}$ when $\mathcal{X}=u=0$ or $\mathsf{E}[\mathcal{Y}|\mathcal{X}=0]$? Which is larger, $\hat{\mathcal{Y}}$ when $\mathcal{X}=u=1$ or $\mathsf{E}[\mathcal{Y}|\mathcal{X}=1]$?
Is $\mathsf{E}[\mathsf{var}(\mathcal{Y}|\mathcal{X}=u)] \le \mathsf{var}(\mathcal{Y})(1-\rho^2_{\mathcal{X},\mathcal{Y}})$ as it should be?

4. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are zero-mean jointly Gaussian random variables with variances σ_1^2 and σ_2^2 respectively and correlation coefficient ρ. The joint pdf surface is a flattened bell. Now suppose we *rotate the coordinate axes* by an angle θ. The random point in the plane whose coordinates are $(\mathcal{X},\mathcal{Y})$ with respect to the old axes now has coordinates $(\mathcal{Z},\mathcal{W})$ with respect to the new axes where $\mathcal{Z}=\mathcal{X}\cos\theta+\mathcal{Y}\sin\theta$ and $\mathcal{W}=\mathcal{Y}\cos\theta-\mathcal{X}\sin\theta$. The joint pdf surface is still the same *shape* as before, but since the coordinate axes have been changes, the pdf surface is defined by a different formula.

   (a) Find the means and variances of the random variables $\mathcal{Z}$ and $\mathcal{W}$, and compute $\mathsf{cov}(\mathcal{Z},\mathcal{W})$. From these values, we can write down the new formula for the joint pdf surface.

   (b) For suitable choice(s) of θ, the new coordinate axes are parallel to the major and minor axes of the ellipses that are the contours of the joint pdf surface. For these choices of θ, $\mathcal{Z}$ and $\mathcal{W}$ are *independent* Gaussian random variables.

   Find an angle θ such that $\mathcal{Z}$ and $\mathcal{W}$ are independent Gaussian random variables. (The other angles are $\theta+\pi/2$, $\theta+\pi$, and $\theta+3\pi/2$).

   You may express your answer as a trigonometric function involving σ_1^2, σ_2^2 , and ρ. In particular, what is the value of θ if $\sigma_1=\sigma_2$?

5. The jointly Gaussian random variables $\mathcal{X}$ and $\mathcal{Y}$ have means 0 and 14 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{16}$.

   (a) Find the pdf of the random variable $\mathcal{Z}=5\mathcal{X}+\mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all α, $-\infty<\alpha<\infty$.

   (b) Find the numerical value of $P\{\mathcal{Y}>3\mathcal{X}\}$.