University of Illinois Fall 2008

ECE 313: Problem Set 12
Joint Distributions.

This Problem Set contains five problems

Due: Wednesday, December 3rd at 4 p.m.
Reading: Ross Ch. 6; PowerPoint Lectures 33-39

1. $\mathcal{X}$ and $\mathcal{Y}$ denote *independent* standard Gaussian random variables.

   (a) What is the joint pdf $f_{\mathcal{X},\mathcal{Y}}(u,v)$ of $\mathcal{X}$ and $\mathcal{Y}$?

   (b) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$. Compute $P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$.

   (c) Let $\mathcal{Z} = \mathcal{X}^2+\mathcal{Y}^2$. What is the pdf of $\mathcal{Z}$?

   (d) Express $P\{|\mathcal{X}| > \alpha\}$ in terms of the complementary unit Gaussian CDF function $Q(x)$, and use this to write $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$ in terms of $Q(x)$. (Remember commas mean intersections).

   (e) On your sketch of part (b), show the region over which you must integrate the joint pdf to find $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$. Use your sketch to prove the following result: $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} < P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$ for $\alpha > 0$.

   (f) Show that inequality of part (e) implies that $Q(x) < \frac{1}{2}\exp(-x^2/2)$ for $x > 0$.

   (g) On your sketch of parts (b) and (d), show the region over which you must integrate to find $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\}$, and prove that

   $$P\{\mathcal{X}^2+\mathcal{Y}^2 \le \alpha^2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2+\mathcal{Y}^2 < 2\alpha^2\}.$$

   Use these inequalities to deduce the *lower* bound $Q(x) > \frac{1}{4}\exp(-x^2)$ for $x > 0$. Note that at $x = 0$, equality holds in the upper bound of part (f) but not in this lower bound .

2. Consider random variables $\mathcal{X}$ and $\mathcal{Y}$ that have the following joint PDF

   $$f_{X,Y}(u,v) = c\sqrt{u^2+v^2}$$

   where u, v belong to the unit circle centered at the origin.

   (a) Find c.

   (b) Using LOTUS, compute $\mathsf{E}[f(X,Y)]$.

   (c) Calculate $P\{f(X,Y) \ge \frac{1}{2}\}$.

3. Random variables $\mathcal{X}$ and $\mathcal{Y}$ have a uniform joint density on the square bounded with corners at the points: $(1,0), (0,1), (-1,0)$ and $(0,-1)$.

   (a) Calculate the marginal pdfs of $\mathcal{X}$ and $\mathcal{Y}$. Are $\mathcal{X}$ and $\mathcal{Y}$ independent?

   (b) Compute $E[\mathcal{X}]$ and $var(\mathcal{X})$.

(c) Calculate the pdf of the random variable $\mathcal{X} + \mathcal{Y}$.

(d) Calculate the pdf of the random variable $\mathcal{X}/\mathcal{Y}$.

4. Define a "random chord" by choosing the midpoint of the chord to be anywhere inside the circle of radius one with equal probability. The chord is, of course, perpendicular to the diameter that passes through the chosen point. Thus, let the random point $(\mathcal{X}, \mathcal{Y})$ be *uniformly distributed* on the interior of the circle of unit radius centered at the origin (this region is called the unit disc).

   (a) Find the probability that the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

   (b) Express $\mathcal{L}$ as a function of the random variable $(\mathcal{X}, \mathcal{Y})$ and find the probability density function for $\mathcal{L}$.

   (c) Find the average length of the chord, i.e. find $\mathsf{E}[\mathcal{L}]$.

5. If the random variables $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are independent, and chosen uniformly in the interval $[0, 1]$, what is the probability that all of the roots of the equation $\mathcal{X}x^2 + \mathcal{Y}x + \mathcal{Z} = 0$ are real?