University of Illinois Fall 2008

ECE 313: Problem Set 11

Decision-Making; System Reliability; Joint Distributions

This Problem Set contains five problems

Due:	Wednesday, November 19 at 4 p.m.
Reading:	Ross Ch. 6; PowerPoint Lectures 19, 29-34
Noncredit Exercises:	**Chapter 6:** Problems 1, 9-15, 20-23

1. Consider the following decision-making problem. If hypothesis H_0 is true, the continuous random variable $\mathcal{X} \sim U(-2,2)$, while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is

$$f_1(u) = \begin{cases} \frac{1}{4}(2-|u|), & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) The *maximum-likelihood* decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$. Specify whether x denotes 0 or 1, and find the values of η, the probability of false alarm P_{FA}, and the probability of missed detection P_{MD}.

   (b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

   (c) The MAP (also known as the minimum-error-probability or Bayesian) decision rule can be stated in the form $|\mathcal{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether x denotes 0 or 1, and find the values of ξ and the error probability $P(E)$.

   (d) For what range (if any) of values of π_0, does the MAP decision rule always choose hypothesis H_0?

   (e) For what range (if any) of values of π_0, does the MAP decision rule always choose hypothesis H_1?

2. As discussed in Lecture 19 of the Powerpoint slides, the probability of failure of a TMR system with perfect majority gate is $3p^2 - 2p^3$ where p is the probability of failure of each module, and the modules as assumed to fail independently of each other. Now, suppose that the system is put into operation at $t = 0$, and let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ denote the time of failure of each module. The independence of failures enters into our calculations as the assertion that for all $t_1, t_2, t_3 > 0$, the events $\{\mathcal{X}_1 > t_1\}, \{\mathcal{X}_2 > t_2\}, \{\mathcal{X}_1 > t_3\}$ are independent events. Note that the occurrences of these events are equivalent to the assertions that modules 1, 2, 3 respectively have *not* failed (i.e., are operational) at times t_1, t_2, t_3. We model $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ as exponential random variables with parameter λ.

   Let $\mathcal{Y}$ denote the time of failure of the TMR system so that the occurrence of the event $\{\mathcal{Y} > T\}$ means that the TMR system is operational at time T.

   (a) Express the event $\{Y > T\}$ in terms of unions, intersections and complements of the events $\{\mathcal{X}_1 > T\}, \{\mathcal{X}_2 > T\}, \{\mathcal{X}_3 > T\}$.

(b) Show that $P\{\mathcal{Y} > T\} = 3\exp(-2\lambda T) - 2\exp(-3\lambda T)$.

i. Use the above result to find $\mathsf{E}[\mathcal{Y}]$ using $\mathsf{E}[\mathcal{Y}] = \displaystyle\int_0^\infty P\{\mathcal{Y} > T\}\,dT$.

This is the *average lifetime* of the TMR system, also known as the mean time before failure (MTBF) or mean time to failure (MTTF) in the reliability literature.

ii. Use the above result to find the pdf of $\mathcal{Y}$ and the hazard rate $h_{\mathcal{Y}}(t)$ of the TMR system. What is the asymptotic value of $h_{\mathcal{Y}}(t)$ as $t \to \infty$?

(c) Find the *median* value of $\mathcal{Y}$ by solving the equation $P\{\mathcal{Y} > T\} = \frac{1}{2}$ for T.

(d) Compare your answers of parts (b) and (c) to the MTBF λ^{-1} and the median lifetime $\lambda^{-1}\ln 2$ for a single module. Do the answers surprise you? Is the TMR system a more reliable system as claimed?

(e) Now suppose that $\lambda = -\ln 0.999$. What are the numerical values of $P\{\mathcal{X}_1 > 1\}$ and $P\{\mathcal{Y} > 1\}$?

(f) I hope you found in part (e) that $P\{\mathcal{X}_1 > 1\} = 0.999$ and so a single module works with 99.9% reliability for at least one unit of time. What is the largest value of T for which $P\{\mathcal{Y} > T\} \geq 0.999$? How does the TMR system compare to a single module in terms of providing 99.9% reliability over long periods of time?

3. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 0.5, & 0 \leq u < 1, 0 \leq v < 1, \text{ and } 0 \leq u+v < 1, \\ 1.5, & 0 \leq u < 1, 0 \leq v < 1, \text{ and } 1 \leq u+v < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal pdf of $\mathcal{X}$.

(b) Find $P\{\mathcal{X} + \mathcal{Y} \leq 3/2\}$ and $P\{\mathcal{X}^2 + \mathcal{Y}^2 \geq 1\}$.

4. Ross, Problem 8, page 313.

5. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathcal{X},\mathcal{Y}}(u,v)$ is nonzero.

(b) Find the marginal pdfs of $\mathcal{X}$ and $\mathcal{Y}$.

(c) Find $P\{\mathcal{Y} > 3\mathcal{X}\}$.

(d) For $\alpha > 0$, find $P\{\mathcal{X} + \mathcal{Y} \leq \alpha\}$.