University of Illinois Fall 2008

ECE 313: Problem Set 10
Gaussian Random Variables;
Functions of Continuous Random Variables; and Hypothesis Testing

This Problem Set contains 6 problems

Due: November 12 at 4 p.m.
Reading: Ross, Chapter 5; Lecture Notes 27, 28, 29.

1. Let $\mathcal{X}$ be a standard Gaussian random variable.
   (a) Find the moments of $\mathcal{X}$ of odd order, i.e., find $E\left[\mathcal{X}^{2k+1}\right]$, for $k = 1, 2, \ldots$.
   (b) Show that the $2k$-th moment of $\mathcal{X}$ is equal to
   $$E\left[\mathcal{X}^{2k}\right] = \frac{(2k)!}{2^k\, k!}.$$
   (c) Suppose that $\mathcal{X}$ is a voltage. Find the pdf of the power dissipated by an R-ohm resistor with this voltage, i.e. $P = \mathcal{X}^2/R$.

2. Let $\Phi(x)$ and $\phi(x)$ be the cumulative distribution function and density, respectively, of the standard normal distribution. In this problem, you are asked to prove the bounds for $1 - \Phi(x)$ given in class. Note that for $x > 0$,
   $$\Phi(-x) = 1 - \Phi(x) = \int_{-\infty}^{-x} \frac{1}{t}\, t\, \phi(t)\, dt,$$
   (a) Integrate the above integral by parts in order to show that $1 - \Phi(x) < \frac{1}{x}\,\phi(x)$, for $x > 0$.
   (b) Apply the same steps as in part (a) to the inequality above and show that $1 - \Phi(x) > \left(\frac{1}{x} - \frac{1}{x^3}\right)\phi(x)$, for $x > 0$.

3. The probability density function of the lifetime $\mathcal{X}$ (measured in hours) of a certain type of electronic device is exponential with parameter $\lambda = 1$. What is the probability that of six such types of devices at least three will function for at least 15 hours? What assumptions would you like to make in your solution?

4. [Read Example 3d (pp. 217-218) in Chapter 5 of Ross first] Let the straight line segment ACB be a diameter of a circle of unit radius and center C. Consider an *arc* AD of the circle where the length $\mathcal{X}$ of the arc (measured clockwise around the circle) is a random variable uniformly distributed on $[0, 2\pi)$. Now consider the *random chord* AD whose length we denote by $\mathcal{L}$.
   (a) Find the probability that $\mathcal{L}$ is greater than the side of the equilateral triangle inscribed in the circle.
   (b) Express $\mathcal{L}$ as a function of the random variable $\mathcal{X}$, and find the pdf for $\mathcal{L}$.

5. If hypothesis H_0 is true, the pdf of $\mathcal{X}$ is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is exponential with parameter 10.

(a) Sketch the two pdfs.

(b) State the *maximum-likelihood* decision rule in terms of a threshold test on the *observed value* u of the random variable $\mathcal{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ to 1.

(c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b)?

(d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_0/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathcal{X}$.

(e) If $\pi_0 = 1/3$, what is the *average* probability of error of the Bayesian decision rule?

(f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by $\mathcal{X}$?

(g) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathcal{X}$. What is the average probability of error for the maximum-likelihood rule when $\pi_0 > 2/3$?

6. Consider the following hypothesis testing problem:

$$\begin{aligned} \mathsf{H}_0 &: X \sim \mathcal{N}(0, \sigma_0^2) \\ \mathsf{H}_1 &: X \sim \mathcal{N}(0, \sigma_1^2), \end{aligned}$$

where $\sigma_0^2 = 1$ and $\sigma_1^2 = 2$.

(a) Show that both the Bayes decision rule and the ML decision rule simplify to comparing $|X|$ to a threshold. Specify that threshold in both cases. (*hint: consider the log likelihood ratio*).

(b) Calculate the false alarm and missed detection probabilities for the ML decision rule.