University of Illinois Fall 2008

$\mathsf{ECE\,313}$: Problem Set 9

Continuous Random Variables; LOTUS; Functions of RVs

This Problem Set contains seven problems

Due:	Wednesday, November 5 at 4 p.m.
Reading:	Ross Ch. 5; PowerPoint Lectures 22-27
Noncredit Exercises:	**Chapter 5:** Problems 13-19; 21, 22, 24, 31-41

1. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable X with probability density function

$$f_{\mathcal{X}}(u) = \begin{cases} 5(1-u)^4, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

   (a) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.

   (b) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

   (c) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathcal{Y}$ denote the amount of gasoline sold per week. How is $\mathcal{Y}$ related to $\mathcal{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit?

   (d) Suppose that the owner pays \$$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly net profit and what value of C maximizes the average weekly net profit?

2. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.

   (a) If $\mathcal{Y} = \mathcal{X}^2$, what are the mean and variance of $\mathcal{Y}$?

   (b) If $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \ge 0 \\ -u^2, & u < 0 \end{cases}$ use LOTUS to find $\mathsf{E}[Z]$.

3. Let $\mathcal{X}$ denote a Gaussian random variable with mean -10 and variance $\sigma^2 = 4$. You have at your disposal two calculators: one can calculate $\Phi(x)$ for $x \ge 0$, and the other can calculate $Q(x)$ for $x \ge 0$. Both have the usual assortment of basic arithmetic functions. Write down expressions for calculating each of the following probabilities with each of the calculators. Remember that the argument of Φ or Q must be ≥ 0 in all cases.
   (**a**) $P\{\mathcal{X} < 0\}$. (**b**) $P\{-10 < \mathcal{X} < 5\}$. (**c**) $P\{\,|\mathcal{X}| \ge 5\}$. (**d**) $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\}$.

4. The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

(a) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

(b) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

5. The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V) - 1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf
$$f_{\mathcal{V}}(u) = 0.5 \exp(-|u|), -\infty < u < \infty.$$
Then, the current $\mathcal{I}$ is also a continuous random variable.

   (a) What values can $\mathcal{I}$ take on?
   (b) Find the CDF of $\mathcal{I}$.
   (c) Find the pdf of $\mathcal{I}$.

6. The time of arrival of a radar echo at a radar receiver can be modeled as a continuous random variable taking on positive values. However, because much of the receiver is implemented using digital circuits (with clock signals), the time of arrival is recorded as $\mathcal{Y} = \lceil \mathcal{X} \rceil$, that is, rounded up to the time of the next clock pulse after the echo is received. Note that $\mathcal{Y}$ is a discrete random variable taking on values $1, 2, \ldots$.

   (a) Express $P\{\mathcal{Y} = k\}$ in terms of the CDF $F_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$.
   (b) Now, suppose that $\mathcal{X}$ is an exponential random variable with parameter λ and CDF $1 - \exp(-\lambda u)$ for $u \geq 0$. Show that $\mathcal{Y}$ is a *geometric* random variable.
   (c) For $\mathcal{X}$ as specified in part (b), show that if λ is small, then $\mathsf{E}[\mathcal{Y}] \approx \mathsf{E}[\mathcal{X}] = \lambda^{-1}$ while if λ is very large, then $\mathsf{E}[\mathcal{Y}]$ approaches 1 while $\mathsf{E}[\mathcal{X}] = \lambda^{-1}$ approaches 0. Give an intuitive explanation justifying this observation.

7. ["Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!"] A signal $\mathcal{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathcal{Y}$ (where $\mathcal{Y} = \alpha$ if $\mathcal{X} > 0$ and $\mathcal{Y} = -\alpha$ if $\mathcal{X} \leq 0$) is used. Note that $\mathcal{Y}$ is a *discrete* random variable.

   (a) What is the pmf of $\mathcal{Y}$?
   (b) The *squared error* in representing $\mathcal{X}$ by $\mathcal{Y}$ is $\mathcal{Z} = \begin{cases} (\mathcal{X} - \alpha)^2, & \text{if } \mathcal{X} > 0, \\ (\mathcal{X} + \alpha)^2, & \text{if } \mathcal{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathcal{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathcal{Z}]$. Use LOTUS to ez-ily calculate $\mathsf{E}[\mathcal{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathcal{Z}]$.
   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathcal{X}$ to the nearest integer $\mathcal{W}$ in the range 3 to +3. Thus, $\mathcal{W} = 3$ if $\mathcal{X} \geq 2.5$, $\mathcal{W} = 2$ if $1.5 \leq \mathcal{X} < 2.5$, $\mathcal{W} = 1$ if $0.5 \leq \mathcal{X} < 1.5$, $\cdots$, $\mathcal{W} = -3$ if $\mathcal{X} < -2.5$. Note that $\mathcal{W}$ is also a discrete random variable. Find the pmf of $\mathcal{W}$.
   (d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathcal{W}$. Suppose that the output is $(\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0)$. What is the pmf of $\mathcal{Z}_2$? the pmf of $\mathcal{Z}_1$? the pmf of $\mathcal{Z}_0$? Note that $(1, 0, 0)$ which represents -4 is not one of the possible outputs from this A/D converter.