University of Illinois Fall 2008

ECE 313: Problem Set 6

Law of Total Probability, Bayes' Formula, Independent Events

This Problem Set contains six problems

Due:	Wednesday, October 15th at 4 p.m.
Reading:	Ross Ch. 3; PowerPoint Lectures 12-14
Noncredit Exercises:	**Chapter 3:** Problems 53, 58, 59, 62, 63, 66, 70-74, 78, 81
	Theoretical Exercises 6, 9, 11 25, 28; Self-Test Problems 11-21

Reminder: Hour Exam I is scheduled for Monday October 13, 7:00 p.m. – 8:00 p.m. in **Room 1404 Siebel Center**. Note the location; the exam was scheduled in 151 Everitt previously.

One 8.5"×11" sheet of notes is permitted, but the exam is closed-book, closed-notes otherwise. Electronic devices – calculators, PDAs, laptops, cell phones, etc. – are not permitted.

Everything covered in class through the lecture of Wednesday October 8 is included on the exam *except* confidence intervals. Note that maximum-likelihood estimation is included on the Hour Exam. Note also that the Law of Total Probability and Bayes' formula are included but independent events (Section 3.4 of Ross) are not. Thus, working on Problems 1-5 of this Problem Set will help you prepare for the Hour Exam.

If you have a conflict on the exam, please send e-mail no later than October 8 to `sarwate@illinois.edu` specifying what the conflict is. The conflict exam is usually scheduled for 5:45 p.m. – 6:45 p.m., or 8:15 p.m. – 9:15 p.m. that same evening.

1. A magnetic tape storing information in binary form has been corrupted, so that the detected readout signal can contains errors. A stored ZERO is read out as a ZERO with probability 0.9 and as a ONE with probability 0.1, while a stored ONE is read out as a ONE with probability 0.85 and as a ZERO with probability 0.15. Each stored digit is equally likely to be a ONE or a ZERO.

   (a) What is the probability of reading out a ONE?

   (b) Given that a ONE has been read out, what is the probability that the stored digit was a ONE?

2. ["Nobel, Nobel . . . ": the chorus of a Christmas carol usually sung in October]
   Two different tests A and B are available for detecting the presence of HIV virus in human blood. Test A is *more sensitive* than Test B: when the HIV virus is present (event $H+$), Test A gives a positive result (event $T+$) with probability 0.999 whereas Test B gives a positive result with probability only 0.99. On the other hand, Test B is *more specific* than Test A: when the HIV virus is *not* present (event $H-$), Test B gives positive result $T+$ with probability only 0.001 whereas Test A gives positive result $T+$ with larger probability 0.01. For obvious reasons, such positive results are called *false positives.* False positive results are a huge nuisance in medical testing

(and testing in general) because they can lead to the patient receiving unnecessary treatment. In HIV testing, false positive results can also lead to undesirable familial, social, and societal consequences.. *False negative* results (event $T-$ occurring when the virus is actually present) also have undesirable consequences in that a patient may not get treatment for the disease, and may also unknowingly infect others.

The data above can be summarized as follows:

Test A: $P(T+|H+) = 0.999$ $P(T+|H-) = 0.01$.
Test B: $P(T+|H+) = 0.99$ $P(T+|H-) = 0.001$.

(a) Suppose that 2% of the population has HIV virus in the bloodstream, that is, $P(H+) = 0.02$. Find the value of $P(T+)$ for each test.

(b) For each test, compute $P(H+|T+)$.

(c) For each test, compute $P(H+|T-)$.

3. Let $\mathcal{X}$ and $\mathcal{Y}$ denote two discrete random variables taking on values $1, 2, 3$. $\mathcal{X}$ denotes a number that we wish to transmit over a channel using one of the three signals s_1, s_2 and s_3. Let $s_\mathcal{X}$ denote the signal that is transmitted. Noise in the channel can corrupt the signal, and thus it is possible that the received signal $s_\mathcal{Y}$ is not the same as the transmitted signal $s_\mathcal{X}$. In particular, the *transition matrix* below gives the (conditional) probability that the receiver hears $s_\mathcal{Y}$ when the transmitter sends $s_\mathcal{X}$.

Transmitted	Received $\mathcal{Y}$		
$\mathcal{X}$	1	2	3
1	0.8	0.1	0.1
2	0.05	0.9	0.05
3	0.15	0.05	0.8

For example, this table is saying that a transmitted s_1 is received as an s_1, or s_2 or s_3 with probabilities $0.8, 0.1$, and 0.1 respectively.

(a) Suppose that $\mathcal{X}$ has pmf $p_\mathcal{X}(1) = 0.5$, $p_\mathcal{X}(2) = 0.25$, $p_\mathcal{X}(3) = 0.25$. What is the pmf of $\mathcal{Y}$?

(b) Given that the receiver heard $\mathcal{Y} = 3$, what are the *conditional* probabilities of $\{\mathcal{X} = 1\}$? $\{\mathcal{X} = 2\}$? $\{\mathcal{X} = 3\}$?

4. A mobile station (MS) is in one of four disjoint cells, numbered 1 through 4. When the MS must be found, it is paged in one cell at a time. Due to channel fading, whenever the MS is paged in the right cell, the page is successful with probability 0.9, and otherwise the page is a miss. The MS is first paged in cell 1. If it is not found there (this happens if the MS wasn't in cell 1, or if it was in cell 1 and the first page was a miss), it is paged in cell 2. If it is not found there, it is paged in cell 3. If it is not found there, it is paged in cell 4. If the MS is not found after all four pages, a second round of pages is started. If the MS hasn't been found after a second round of pages, the overall search is a failure.

Let E_i denote the event that the mobile station is located in cell i, and suppose that $P(E_i) = \dfrac{5-i}{10}$ for $1 \le i \le 4$.

Let F_k be the event that the MS is found on the k^{th} page, and let G_k be the event that the MS is not found within the first k pages, for $1 \le k \le 8$.

(a) What is $P(F_1)$?

(b) What is $P(E_1|F_1)$?

(c) Find $P(G_8)$, which is the probability that the overall search is a failure.

(d) Find $P(F_2|G_1)$.

(e) Find $P(F_4|G_3)$.

(f) Find $P(F_8|G_7)$.

5. With probability 0.8, Adam has committed the crime for which he is about to be tried. Bo and Clara, each of whom knows whether or not Adam has committed the crime, have been called to testify. Bo likes Adam and will tell the truth if Adam did not commit the crime but will lie with probability 0.2 if Adam did commit the crime. Clara does not like Adam and will tell the truth if Adam committed the crime but will lie with probability 0.3 if Adam did not commit the crime.

   (a) Who is more likely to testify truthfully at the trial, Bo or Clara?

   (b) What is the probability that Bo and Clara give conflicting testimony?

   (c) What is the conditional probability that Adam did not commit the crime given that Bo and Clara gave conflicting testimony at the trial?

   In the interest of easing the grader's burden, please use A to denote the event that Adam committed the crime, B to denote the event that Bo testifies truthfully, and C to denote the event that Clara testifies truthfully.

6. An experiment consists of three independent tosses of a fair coin. Let A and B respectively denote the events that the first and second tosses result in a Head, and let C denote the event that exactly two Heads occur *and* they occur on two successive tosses.

   (a) Are A and C independent events?

   (b) Are B and C independent events?

   (c) Are A, B, and C independent events?

   (d) Are A, B, and C *pairwise* independent events?

   SHOW YOUR WORK!