University of Illinois Fall 2008

ECE 313: Problem Set 5
Maximum-Likelihood Estimation; Conditional Probability

This Problem Set contains seven problems

Due:	Wednesday October 8, 4:00 p.m.
Reading:	Ross, Chapter 3, Sections 1-3
	Powerpoint slides, Lectures 9 and 10
Noncredit Exercises:	DO NOT turn these in.
	Chapter 3: Problems 1, 2, 5, 10, 12, 16, 31, 38, 39, 44
	Theoretical Exercises 1, 2, 8; Self-Test Problems 1-10

Reminder: Hour Exam I is scheduled for Monday October 13, 7:00 p.m. – 8:00 p.m. in **Room 1404 Siebel Center**. Note the location; the exam was scheduled in 151 Everitt previously.

One 8.5"×11" sheet of notes is permitted, but the exam is closed-book, closed-notes otherwise. Electronic devices – calculators, PDAs, laptops, cell phones, etc. – are not permitted.

If you have a conflict on the exam, please send e-mail no later than October 8 to `sarwate@illinois.edu` specifying what the conflict is. The conflict exam is usually scheduled for 5:45 p.m. – 6:45 p.m., or 8:15 p.m. – 9:15 p.m. that same evening.

1. Let $\mathcal{X}$ denote a discrete random variable that takes on integer values $1, 2, \ldots, n$. The value of n is unknown, and we wish to find its maximum-likelihood estimate $\hat{n}_{\rm ML}$ from the observation that $\mathcal{X}$ had value 10 on a trial of the experiment.

   (a) Explain why $\hat{n}_{\rm ML}$ must be 10 or more.

   (b) Suppose that $\mathcal{X}$ has the *increasing-ramp* pmf $p_{\mathcal{X}}(k) = \begin{cases} \frac{2k}{n(n+1)}, & 1 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$

   What is $\hat{n}_{\rm ML}$ in this case?

   (c) Suppose that $\mathcal{X}$ has the *decreasing-ramp* pmf $p_{\mathcal{X}}(k) = \begin{cases} \frac{2(n+1-k)}{n(n+1)}, & 1 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$

   i. Compute the value of $p_{\mathcal{X}}(10)$ for $n = 10, 11, 12, \ldots$ and find the maximum-likelihood estimate $\hat{n}_{\rm ML}$ numerically.

   ii. Now suppose that $\mathcal{X}$ has value i. Find $\hat{n}_{\rm ML}$ as a function of i and verify that when $i = 10$, your function gives the same value for $\hat{n}_{\rm ML}$ as you found in part (c)(i).

2. There are N multiple-choice questions (with 5 possible answers each) on a certain exam. A student knows the answers to K questions and answers them correctly. On the remaining $N - K$ questions, the student guesses randomly among the 5 choices. The examiner knows N, and can observe the values of $\mathcal{C}$, the number of *c*orrect answers, and $\mathcal{W} = N - \mathcal{C}$, the number of *w*rong answers on the answer sheet. Note

that $\mathcal{C}$ can have values K, $K+1$, $\ldots$, N. What the examiner is really interested in, though, is *estimating* the value of K.

(a) Explain why it is reasonable to model $\mathcal{W}$ as a binomial random variable with parameters $(N-K, 0.8)$. What assumptions are you making?

(b) Suppose that n answers are incorrect, that is, $\mathcal{W}=n$ and $\mathcal{C}=N-n$. What is the *likelihood* of this observation? Hint: your answer will depend on N, n and the unknown parameter K that the examiner is interested in estimating.

(c) Having observed that $\mathcal{W}=n$, the examiner is sure that K cannot exceed $N-n$, i.e., K can have value $0,1,2,\ldots,N-n$ only. Use the method of Proposition 6.1 of Chapter 4 in Ross to show that the likelihood you found in part (b) is maximized at $\hat{K}_{\mathrm{ML}} = \lfloor N - 1.25n + 1 \rfloor$.

(d) Since $\mathcal{C} = N-n$, a *guessing penalty* is applied by subtracting $\lfloor 0.25n \rfloor$ from $\mathcal{C}$ to get an estimate of K. For $N=100$ and $K=90$, compare the *guessing-penalty estimate* $\hat{K}_{\mathrm{GP}} = N - n - \lfloor 0.25n \rfloor$ and the maximum likelihood estimate $\hat{K}_{\mathrm{ML}}$ for each possible value that n can take on, *viz.* $n = 0, 1, \ldots 10$. Notice that lucky guesses cause the examiner to overestimate K while the unlucky student who blows all ten problems has to suffer the further indignity of having the score reduced to something smaller than K.

3. [Polya's urn scheme] An urn contains r red and g green balls. Two balls are drawn at random from the urn, with the first ball being returned to the urn (which is then well shaken) before the second ball is drawn. Let R_1 and R_2 respectively denote the events that the first and second balls are red.

   (a) What are $P(R_1)$ and $P(R_2)$?

   (b) Now suppose that when the first ball is returned to the urn, c *additional* balls of the *same color* are also put into the urn (which is then well shaken before the second ball is drawn.) Clearly $P(R_1)$ is the same as before, but what is $P(R_2)$ now? Remember that the urn now contains $r+g+c$ balls. Simplify your answer and compare to the value of $P(R_2)$ that you obtained in part (a).

   (c) For the experiment of part (b), what is the conditional probability that the urn contained $r+c$ red balls given that R_2 occurred?

4. ["Take me out to the ball game...] A baseball pitcher's repertoire is limited to *fastballs* (event F), *curve balls* (event C) and *sliders* (event S). It is known that $P(C) = 2P(F)$. Whether the event H that the batter hits the ball occurs depends on the pitch, and it is known that $P(H|F) = 2/5, P(H|C) = 1/4$, and $P(H|S) = 1/6$.

   If $P(H) = 1/4$, what is $P(C)$?

5. This problem on conditional probability has three unrelated parts:

   (a) If $P(A|B) = 0.3, P(A^c|B^c) = 0.4$, and $P(B) = 0.7$, find $P(A|B^c), P(A)$, and $P(B|A)$.

   (b) If $P(E) = \frac{1}{4}, P(F|E) = \frac{1}{2}$, and $P(E|F) = \frac{1}{3}$, find $P(F)$.

   (c) If $P(G) = P(H) = \frac{2}{3}$, show that $P(G|H) \geq \frac{1}{2}$.

6. Monty Hall, the host of the TV game show "Let's Make A Deal" shows you three curtains. One curtain conceals a car, while the other two conceal goats. All three curtains are equally likely to conceal the car. He offers you the following "deal": pick a curtain, and you can have whatever is behind it. When you pick a curtain, instead of giving you your just deserts, Monty (who knows where the car is) opens one of the remaining curtains to show you that there is a goat behind it, and offers the following "new, improved deal" : you can either stick with your original choice, or switch to the remaining (unopened) curtain. Amidst the deafening roars of "Stand pat" and "Switch, you idiot" from the crowd, Monty points out that previously your chances of winning were 1/3. Now, since you know that the car is behind one of the two unopened curtains, your chances of winning have increased to 1/2, and thus the new improved deal is indeed better.

   (a) Let A denote the event that your first choice of door has the car behind it. What is $P(A)$?

   Let B denote the event that your second choice of door has the car behind it.

   (b) *Your strategy is to always stay put* and so your second choice is the same as your first choice. What is $P(B|A)$ in this case? What is $P(B|A^c)$? Use these results to find $P(B)$ for the stay-put strategy.

   (c) *Your strategy is to always switch* and so your second choice is the other unopened door. What is $P(B|A)$ in this case? What is $P(B|A^c)$? Use these results to find $P(B)$ for the always-switch strategy.

   (d) *Your strategy is to pick randomly* and so your second choice is equally likely to be either unopened door. What is $P(B|A)$ in this case? What is $P(B|A^c)$? Use these results to find $P(B)$ for the pick-randomly strategy. Is Monty correct in asserting that if you choose randomly between the two unopened curtains, you have a probability of winning of 1/2?

   (e) Having disposed of your goat, you return the next day to the show, and this time, Monty calls you *and* your friend to come on down and choose one curtain each. Which is better: to be the first to pick a curtain or the second? Or does it not make a difference? This time, Monty opens the curtain chosen by your friend to reveal a goat and sends him back to his seat. He now asks whether you want to stick with your original choice or switch to the the third (unchosen) curtain. Which choice gives you a larger chance of winning the car?

   Note: The rules of the game of parts (a)-(d) are that Monty always opens one of the two unchosen curtains and he always offers the "new improved deal," i.e., he never opens a curtain to reveal the prize (saying "Oops, you lose; return to your seat."). In the game of part (e), he always opens one of the chosen curtains to eliminate one of the contestants and then always offers the other contestant the chance to switch.

7. At the County Fair, you see a man sitting at a table and rapidly rolling a pea between three walnut shells. "Step right up, me bucko, and try your luck! The hand is quicker than the eye!" he says, and hides the pea under one of the shells. You have no idea which shell is covering the pea, but you point to one shell at random and bet that the pea is under it. The man picks up one of the shells that you didn't choose, and shows you that the pea is not underneath that shell. He asks if you would like to switch your bet to the other unchosen shell. Should you accept the offer? Why or why not? How does this game differ from the one analyzed in Problem 6 parts (a)-(d)?