University of Illinois Fall 2008

ECE 313: Problem Set 4

Due:	Wednesday, October 1st, at 4 pm.
Reminder:	No classes on September 24th and September 26th!
Reading:	Ross, Chapter 4
Noncredit Exercises:	DO NOT turn these in.
	Chapter 4: Problems 32, 38-39, 40-43, 47-52;
	Theoretical Exercises 16-19; Self-Test Problems 9, 13, 15, 16.

This Problem Set contains seven problems

1. Use a spreadsheet/Mathematica/MATLAB for this problem.
   Let A denote an event of probability p.

   (a) For $p = 0.1, 0.25, 0.4, 0.5, 0.6, 0.75,$ and 0.9, find the numerical values of the probabilities that A occurs $0, 1, 2, \ldots, 10$ times on 10 trials.

   (b) You have computed the pmf of a binomial random variable $\mathcal{X}_p$ with parameters $(10, p)$ for seven choices of p. For each value of p, draw a bar graph of the pmf of $\mathcal{X}_p$. (The pmf of $\mathcal{X}_{0.5}$ is shown on page 157 of the text!).

   (c) What is the relationship between the pmfs of $\mathcal{X}_p$ and $\mathcal{X}_{1-p}$?

2. Let $\mathcal{X}$ denote a binomial random variable with parameters (N, p).

   (a) Show that $\mathcal{Y} = N - \mathcal{X}$ is a binomial random variable with parameters $(N, 1-p)$.

   (b) What is $P\{\mathcal{X} \text{ is even}\}$? Hint: Use the binomial theorem to write an expression for $(x+y)^n + (x-y)^n$ and then set $x = 1-p$, $y = p$.

3. Eight persons have purchased tickets ($\$F$ per person) for travel in a 5-passenger plane on a scheduled airline flight in Ruritania. The number of persons who actually show up to travel can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(8, 0.5)$. Naturally, if more than 5 persons show up, only 5 get to go and the rest are left behind. Let $\mathcal{Y}$ denote the number of persons left behind.

   (a) What is $\mathsf{E}[\mathcal{X}]$?

   (b) Find the pmf of $\mathcal{Y}$.

   (c) What is $\mathsf{E}[\mathcal{Y}]$? Calculate this in two ways: (i) from your answer to part (b), and (ii) by using the fact that $\mathcal{Y}$ is a function of $\mathcal{X}$, and hence LOTUS allows us to calculate $\mathsf{E}[\mathcal{Y}]$ directly from the pmf of $\mathcal{X}$.

   (d) The flight costs the airline \$200 plus \$10 for each passenger carried on board. (Even if no passengers show up, the flight must still go because the plane is needed at the destination for use in the return flight). According to Ruritanian Aviation Administration (RAA) rules, passengers who don't show up are SOL; they cannot use their tickets on another flight and they cannot get a refund either. On the other hand, each bumped passenger gets a full refund of $\$F$ plus \$20 as compensation for being denied boarding. Let $\mathcal{Z}$ denote the net profit to the airline from this flight. Use LOTUS to express $\mathsf{E}[\mathcal{Z}]$ as a function of F and determine the value of F for which the average profit is exactly 0, i.e. the break-even point.

(e) What is the pmf of $\mathcal{Z}$?

4. An absent-minded professor has n keys in his pocket of which only one (he does not remember which one) fits his office door. He picks a key at random and tries it on his door. If that does not work, he picks a key again to try, and so on till the door unlocks. Let $\mathcal{X}$ denote the number of keys that he tries. Find $\mathsf{E}[\mathcal{X}]$ in the following two cases.

   (a) A key that does not work is put back in his pocket so that when he picks another key, all n keys are equally likely to be picked (sampling with replacement).

   (b) A key that does not work is put in his briefcase so that when he picks another key, he picks at random from those remaining in his pocket (sampling without replacement).

5. Let $\mathcal{X}$ denote a nonnegative integer-valued random variable, and assume that $E[\mathcal{X}]$ is finite.

   (a) Show that $\mathsf{E}[\mathcal{X}] = \sum_{i=0}^{\infty} P\{\mathcal{X} > i\}$ (Note that this is Theoretical Exercise 6, Ross, page 197).

   (b) Find $P\{\mathcal{X} > i\}$ for a geometric random variable $\mathcal{X}$, and then use the result in part (a) to find the expected value of $\mathcal{X}$.

6. Let $\mathcal{Y}$ denote a Poisson random variable with parameter λ.

   (a) Show that $P\{\mathcal{Y} \text{ is even}\} = \exp(-\lambda)\cosh(\lambda)$.

   (b) In Problem 2 of this Problem Set, you proved (I hope!) that the probability that a binomial random variable with parameters (N, p) is *even* is $[1+(1-2p)^N]/2$. Now, for large N and small p, the binomial probability $P\{\mathcal{X} = k\}$ is well approximated by the Poisson probability $\exp(-Np)(Np)^k/k!$. Under the same conditions, show that $[1 + (1 - 2p)^N]/2 \approx \exp(-Np)\cosh(Np)$ and thus your answer of part (a) is consistent with the previous result.

   (c) Now suppose that the value of λ is unknown. The experiment is performed and it is observed that $\mathcal{Y} = k$. What is the *likelihood* of this observation? What is the *maximum likelihood* estimate $\hat{\lambda}$ of λ? That is, what choice of positive number $\hat{\lambda}$ maximizes the likelihood of the observation $\mathcal{Y} = k$?

7. Suppose that 105 passengers hold reservations for a 100-passenger flight from Chicago to Champaign. The number of passengers who show up at the gate can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(105, 0.9)$.

   (a) On average, how many passengers show up at the gate?

   (b) If $\mathcal{X} \leq 100$, everyone who shows up gets to go. Find the value of $P\{\mathcal{X} \leq 100\}$.

   (c) Explain why the number of *no-shows* can be modeled as a binomial random variable $\mathcal{Y}$ with parameters $(105, 0.1)$.

   (d) Notice that the probability that everyone who shows up gets to go can also be expressed as $P\{\mathcal{Y} \geq 5\}$. Use the *Poisson approximation* to compute $P\{\mathcal{Y} \geq 5\}$ and compare your answer to the "more exact" answer that you found in part (b).