University of Illinois Fall 2008

ECE 313: Solutions to the Final Exam

1. **[35 points]** A particular webserver may be working (event W) or not working (event W^c). It is known that $P(W) = 0.8$. An attempt to access the webserver can be a failure (event F) or a success (event F^c). Obviously, $P(F|W^c) = 1$, but even if the webserver is working, an attempt to access it can fail due to network congestion. It is known that $P(F^c|W) = 0.9$.

   (a) **[5 points]** Find $P\{\text{access attempt fails}\}$.
   Solution: $P(F) = P(F|W)P(W) + P(F|W^c)P(W^c) = 0.1 \times 0.8 + 1 \times 0.2 = 0.28$.

   (b) **[10 points]** Find $P\{\text{server is working} \mid \text{access attempt fails}\}$.
   $P(W|F) = \dfrac{P(F|W)P(W)}{P(F)} = \dfrac{0.1 \times 0.8}{0.28} = \dfrac{0.08}{0.28} = \dfrac{8}{28} = \dfrac{2}{7}$.

   Now, suppose that the results of two consecutive attempts to access the webserver are *conditionally independent* events given that the webserver is working, and also *conditionally independent* given that the webserver is not working.

   (c) **[10 points]** Find $P\{\text{second access attempt fails} \mid \text{first access attempt fails}\}$.
   Solution: $P(F_1F_2|W) = P(F_1|W)P(F_2|W) = 0.01$ since F_1 and F_2 are conditionally independent given W. Since F_1 and F_2 are also conditionally independent given W^c, we have that $P(F_1F_2|W^c) = 1$. Hence,

   $$P(F_1F_2) = P(F_1F_2|W)P(W) + P(F_1F_2|W^c)P(W^c) = 0.01 \times 0.8 + 1 \times 0.2 = 0.208,$$

   and $P(F_2|F_1) = \dfrac{P(F_2F_1)}{P(F_1)} = \dfrac{0.208}{0.28} = \dfrac{208}{280} = \dfrac{26}{35}$.

   (d) **[10 points]** Find $P\{\text{server is working} \mid \text{first and second access attempts fail}\}$.
   Solution: $P(W|F_1F_2) = \dfrac{P(F_1F_2|W)P(W)}{P(F_1F_2)} = \dfrac{0.008}{0.208} = \dfrac{8}{208} = \dfrac{1}{26}$.

2. **[20 points]** $\mathcal{X}$ denotes a Poisson random variable with parameter $\ln(3)$. Find the numerical values of the mean and variance of $\mathcal{Y} = \cos(\pi\mathcal{X})$.

   Solution: Since $\mathcal{X}$ takes on integer values, $\mathcal{Y}$ has value $+1$ or -1 according as $\mathcal{X}$ is even or odd, that is, $\mathcal{Y} = (-1)^{\mathcal{X}}$. Hence,
   $$\mathsf{E}[\mathcal{Y}] = \sum_{k=0}^{\infty}(-1)^k \exp(-\lambda)\frac{\lambda^k}{k!} = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = \exp(-2\lambda) = \frac{1}{9} \text{ since } \lambda = \ln(3).$$
   Next, note that $\mathsf{E}[\mathcal{Y}^2] = \sum_{k=0}^{\infty}\exp(-\lambda)\frac{\lambda^k}{k!} = 1$ and hence $\mathsf{var}(\mathcal{Y}) = \mathsf{E}[\mathcal{Y}^2] - (\mathsf{E}[\mathcal{Y}])^2$
   $= 1 - \dfrac{1}{81} = \dfrac{80}{81}$.

3. **[15 points]** For what value of C, if any, is $f(u) = \exp(Cu^2), -\infty < u < \infty$, a valid pdf?

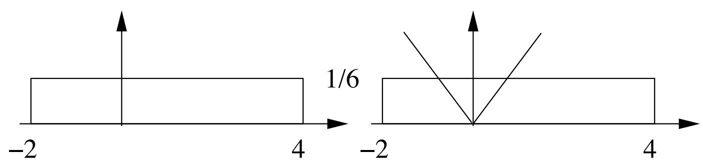
   Solution: $f(u)$ cannot be a valid pdf for $C \geq 0$ since the area under the curve would be ∞, and not 1. For $C < 0$, the function $f(u)$ resembles the pdf $\exp(-u^2/2\sigma^2)/\sigma\sqrt{2\pi}$ of a zero-mean Gaussian random variable with variance σ^2, except for the constant multiplying the exponential. But this constant has value 1 if $\sigma = 1/\sqrt{2\pi}$, and so we see that $f(u) = \exp(-\pi u^2), -\infty < u < \infty$, is the pdf of a zero-mean Gaussian random variable with variance $1/2\pi$.

   In short, $f(u) = \exp(Cu^2), -\infty < u < \infty$, is a valid pdf if and only if $C = -\pi$.

4. **[30 points]** $\mathcal{X}$ denotes a *uniform* random variable with mean 1 and variance 3.

   (a) **[10 points]** Find $P\{\mathcal{X} < 0\}$.

   Solution: A random variable uniformly distributed on $[a, b]$ has mean $\dfrac{a+b}{2}$ and variance $\dfrac{(b-a)^2}{12}$. Hence, we have that $b - a = 6$, and $b + a = 2$, giving $a = -2, b = 4$. The pdf is as shown below. $P\{\mathcal{X} < 0\} = \dfrac{1}{3}$ by inspection; purists can use $\displaystyle\int_{-2}^{0} \frac{1}{6}\,du = \frac{u}{6}\Big|_{-2}^{0} = \frac{1}{3}$.



   (b) **[10 points]** Find $\mathsf{E}[|\mathcal{X}|]$.
   Solution: We use LOTUS to get

$$\mathsf{E}[|\mathcal{X}|] = \frac{1}{6}\left[\int_{-2}^{0} -u\,du + \int_{0}^{4} u\,du\right] = \frac{1}{6}\left[\frac{-u^2}{2}\Big|_{-2}^{0} + \frac{u^2}{2}\Big|_{0}^{4}\right] = \frac{1}{6}[2+8] = \frac{5}{3}.$$

   (c) **[10 points]** Find the pdf of $\mathcal{Y} = |\mathcal{X}|$. In order to receive full credit, you must specify the value of $f_{\mathcal{Y}}(v)$ for all $v, -\infty < v < \infty$.
   Solution: $\mathcal{Y} = |\mathcal{X}|$ takes on values in $[0, 4]$, and hence $F_{\mathcal{Y}}(v) = 0$ for $v < 0$ and $F_{\mathcal{Y}}(v) = 1$ for $v > 4$.
   For any $v, 0 \le v \le 2, F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \le v\} = P\{[-v \le \mathcal{X} \le v\} = v/3$.
   For any $v, 2 \le v \le 4, F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \le v\} = P\{\mathcal{X} \le v\} = (v+2)/6$.
   Hence, $f_{\mathcal{Y}}(v) = \dfrac{d}{du}F_{\mathcal{Y}}(v) = \begin{cases} \frac{1}{3}, & 0 \le v \le 2, \\ \frac{1}{6}, & 2 < v \le 4, \\ 0, & \text{elsewhere.} \end{cases}$
   It is easy to verify that this is a valid pdf, and thus we have not made any obvious errors.

5. **[40 points]** A radio-frequency signal is either a radar echo (hypothesis H_1) or ambient noise (hypothesis H_0). The *phase* of the signal is modeled as a continuous random variable $\mathcal{X}$ whose pdf is as follows:

   - When H_0 is true, $\mathcal{X}$ has pdf $f_0(u) = \begin{cases} \frac{1}{2\pi}, & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$
   - When H_1 is true, $\mathcal{X}$ has pdf $f_1(u) = \begin{cases} \frac{1}{2\pi}\big(1 + \cos u\big), & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$

   The radar receiver measures $\mathcal{X}$ and decides which hypothesis is true.

   (a) **[10 points]** Suppose that the *maximum-likelihood* decision rule is being used. What value(s) of $\mathcal{X}$ result in a decision in favor of H_1?

   Solution: The likelihood ratio is $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = 1 + \cos u > 1$ if and only if $\mathcal{X} \in (-\pi/2, \pi/2) = \Gamma_1$.

   (b) **[10 points]** Find the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

Solution: $P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi} du = \frac{1}{2}$.

$P_{\text{MD}} = 1 - \int_{\Gamma_1} f_1(u)\,du = 1 - \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi}(1+\cos u)\,du = \frac{1}{2} - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos u\,du = \frac{1}{2} - \frac{1}{\pi}$.

(c) [**10 points**] Now suppose that $P(\mathsf{H}_0) = \pi_0 = \frac{1}{3}$, $P(\mathsf{H}_1) = \pi_1 = \frac{2}{3}$. What is the *average* error probability $\bar{P}_e$ of the maximum *a posteriori* probability (MAP) (that is, minimum-error-probability or Bayesian) decision rule?

Solution:

In this case, $\Gamma_1 = \{u : \Lambda(u) > \pi_0/\pi_1\} = \{u : 1 + \cos u > 1/2\}$ $= \{u : \cos u > -1/2\} = (-2\pi/3, 2\pi/3)$ and hence

$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-2\pi/3}^{2\pi/3} \frac{1}{2\pi}\,du = \frac{2}{3}$ while

$P_{\text{MD}} = 1 - \int_{-2\pi/3}^{2\pi/3} \frac{1}{2\pi}(1+\cos u)\,du = \frac{1}{3} - \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} \cos u\,du = \frac{1}{3} - \frac{1}{2\pi}\sin u\Big|_{-2\pi/3}^{2\pi/3} = \frac{1}{3} - \frac{\sqrt{3}}{2\pi}$.

Therefore, $\bar{P}_e = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{2}{3}\left[\frac{1}{3} - \frac{\sqrt{3}}{2\pi}\right] = \frac{4}{9} - \frac{1}{\pi\sqrt{3}}$.

(d) [**10 points**] For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_0 regardless of the value of $\mathcal{X}$?

Solution: $\Lambda(u) = 1 + \cos u$ has maximum value 2 and is thus always smaller than the threshold $\frac{\pi_0}{\pi_1} = \frac{\pi_0}{1-\pi_0}$ if $\pi_0 > \frac{2}{3}$.

6. [**35 points**] The joint pdf of random variables $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} u+v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) [**15 points**] Find $P\{\mathcal{X} + \mathcal{Y} > 1\}$.

Solution:

$$\begin{aligned} P\{\mathcal{X} + \mathcal{Y} > 1\} &= \int_{v=0}^{1}\int_{u=1-v}^{1} u + v\,du\,dv = \int_{v=0}^{1} \frac{u^2}{2} + vu\Big|_{u=1-v}^{1} dv \\ &= \int_{v=0}^{1} \frac{1-(1-v)^2}{2} + v^2 dv = \frac{v}{2} + \frac{(1-v)^3}{6} + \frac{v^3}{3}\Big|_0^1 = \frac{2}{3}. \end{aligned}$$

(b) [**20 points**] Find $P\{\mathcal{X}^2 + \mathcal{Y}^2 \le 1\}$.

Solution:

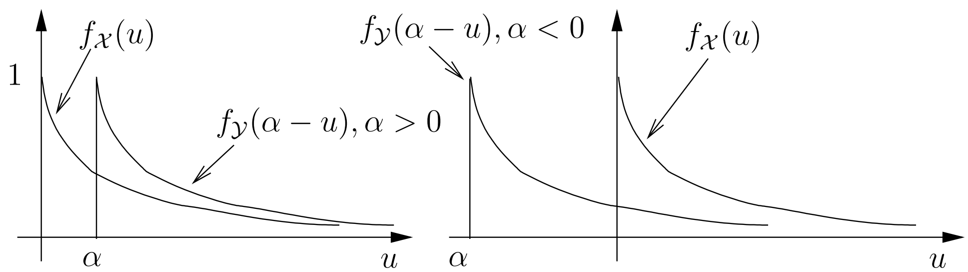
$$\begin{aligned} P\{\mathcal{X}^2 + \mathcal{Y}^2 \le 1\} &= \iint_{u^2+v^2\le 1} f_{\mathcal{X},\mathcal{Y}}(u,v)\,du\,dv = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} (r\cos\theta + r\sin\theta)\,r\,d\theta\,dr \\ &= \int_{r=0}^{1} 2r^2\,dr = \frac{2r^3}{3}\Big|_{r=0}^{1} = \frac{2}{3} \text{ also!} \end{aligned}$$

7. **[30 points]** $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with pdfs as specified below:

$$f_{\mathcal{X}}(u) = \begin{cases} \exp(-u), & u > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\mathcal{Y}}(v) = \begin{cases} \exp(v), & v < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{\mathcal{Z}}(\alpha)$, the pdf of the random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all real numbers α.

Solution: Since $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables, $f_{\mathcal{Z}} = f_{\mathcal{X}} \star f_{\mathcal{Y}}$, that is, $f_{\mathcal{Z}}(\alpha) = \int_{-\infty}^{\infty} f_{\mathcal{X}}(u) f_{\mathcal{Y}}(\alpha - u)\, du = \int_{0}^{\infty} \exp(-u) f_{\mathcal{Y}}(\alpha - u)\, du$ since $f_{\mathcal{X}}(u) = 0$ for $u < 0$. Now, $f_{\mathcal{Y}}(\alpha - u) = 0$ if $\alpha - u > 0$, that is, if $u < \alpha$. As illustrated in the figure below, we have to consider two cases.



- If $\alpha > 0$, then the integrand is 0 for $0 < u < \alpha$, and we get $f_{\mathcal{Z}}(\alpha) = \int_{\alpha}^{\infty} \exp(-u)\exp(\alpha - u)\, du = \exp(\alpha)\frac{1}{2}\exp(-2u)\Big|_{\alpha}^{\infty} = \frac{1}{2}\exp(-\alpha)$.
- If $\alpha < 0$, then the integrand is nonzero for $0 < u < \infty$, and we get $f_{\mathcal{Z}}(\alpha) = \int_{0}^{\infty} \exp(-u)\exp(\alpha - u)\, du = \exp(\alpha)\frac{1}{2}\exp(-2u)\Big|_{0}^{\infty} = \frac{1}{2}\exp(\alpha)$.

We can combine these two cases and write $f_{\mathcal{Z}}(\alpha) = \frac{1}{2}\exp(-|\alpha|), -\infty < \alpha < \infty$.

8. **[20 points]** $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = 1$. Suppose that $\mathcal{Q} = 2\mathcal{X} + 3\mathcal{Y} + 4$ and $\mathcal{R} = 3\mathcal{X} - 4\mathcal{Y} + 2$ and that $\mathsf{var}(\mathcal{Q}) = \mathsf{var}(\mathcal{R})$.

   (a) **[10 points]** What is the value of the *correlation coefficient* $\rho_{\mathcal{X},\mathcal{Y}}$ and the value of $\mathsf{var}(\mathcal{Q})$?
   Solution:

$$\begin{aligned} \mathsf{var}(\mathcal{Q}) &= \mathsf{var}(2\mathcal{X} + 3\mathcal{Y} + 4) = 2^2\mathsf{var}(\mathcal{X}) + 3^2\mathsf{var}(\mathcal{Y}) + 2 \cdot 2 \cdot 3\mathsf{cov}(\mathcal{X}, \mathcal{Y}) \\ &= 13 + 12\mathsf{cov}(\mathcal{X}, \mathcal{Y}). \\ \mathsf{var}(\mathcal{R}) &= \mathsf{var}(3\mathcal{X} - 4\mathcal{Y} + 2) = 3^2\mathsf{var}(\mathcal{X}) + 4^2\mathsf{var}(\mathcal{Y}) - 2 \cdot 3 \cdot 4\mathsf{cov}(\mathcal{X}, \mathcal{Y}) \\ &= 25 - 24\mathsf{cov}(\mathcal{X}, \mathcal{Y}), \end{aligned}$$

   Since $\mathsf{var}(\mathcal{Q}) = \mathsf{var}(\mathcal{R})$, we deduce that $\mathsf{cov}(\mathcal{X}, \mathcal{Y}) = \rho_{\mathcal{X},\mathcal{Y}}\sqrt{\mathsf{var}(\mathcal{X})\mathsf{var}(\mathcal{Y})} = \rho_{\mathcal{X},\mathcal{Y}} = \frac{1}{3}$ and $\mathsf{var}(\mathcal{Q}) = \mathsf{var}(\mathcal{R}) = 17$.

   (b) **[10 points]** Now suppose also that $\mathcal{X}$ and $\mathcal{Y}$ are *jointly Gaussian* random variables. Are $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ *independent* random variables?
   Solution: $\mathsf{cov}(\mathcal{X} + \mathcal{Y}, \mathcal{X} - \mathcal{Y}) = \mathsf{var}(\mathcal{X}) - \mathsf{var}(\mathcal{Y}) = 0$. Hence, $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ are uncorrelated random variables. But since they are also *jointly Gaussian* random variables, $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ are independent random variables.