# ECE 313: Solutions to Hour Exam I

1. A store in the Paynless Shoe Sores chain has 13 different pairs of shoes in a barrel at the door. Suppose that **two** shoes are **picked at random** from the barrel. Note that this is sampling *without* replacement.

   (a) What is the probability of getting a **matching pair** of shoes? What is the probability of getting **two left** shoes? What is the probability of getting a **left** shoe and a **right** shoe but **not** a **matching** pair of shoes?
   **Solution:** This problem can be solved in at least two different ways.
   First suppose the shoes are picked one after another. Then, it does not matter which shoe is the first one; the probability that the second shoe drawn is the mate of the first is exactly $\frac{1}{25}$.
   Next, the probability of getting a left shoe on the first draw is $\frac{13}{26} = \frac{1}{2}$, and given this, the probability of getting a left shoe on the second draw is $\frac{12}{25}$. Hence, the probability of getting two left shoes is $\frac{1}{2} \times \frac{12}{25} = \frac{6}{25}$.
   Finally, to find the probability of getting a left and a right shoe but not a pair, note that it does not matter whether the first shoe drawn is a left shoe or a right shoe. The probability that the second shoe is for the other foot, but not the mate of the first, is $\frac{12}{25}$.
   Alternatively, we are getting a subset of size 2 from the barrel, and so there are $\binom{26}{2}$ equally likely outcomes of the experiment. Exactly 13 of the outcomes are pairs. Hence,
   $P\{\text{matching pair}\} = \dfrac{13}{\binom{26}{2}} = \dfrac{13}{\frac{26\times 25}{1\times 2}} = \dfrac{1}{25}$ as before.
   $P\{\text{two left shoes}\} = \dfrac{\binom{13}{2}}{\binom{26}{2}} = \dfrac{\frac{13\times 12}{1\times 2}}{\frac{26\times 25}{1\times 2}} = \dfrac{6}{25}$ which obviously is also $P\{\text{two right shoes}\}$.
   $P\{\text{one left, one right, not matching}\} = 1 - \dfrac{1}{25} - \dfrac{6}{25} - \dfrac{6}{25} = \dfrac{12}{25}$.
   Another way of getting the result is to note that the probability of getting a left shoe and a right shoe is $P\{\text{one left, one right}\} = \dfrac{\binom{13}{1}\binom{13}{1}}{\binom{26}{2}} = \dfrac{13\times 13}{\frac{26\times 25}{1\times 2}} = \dfrac{13}{25}$ from which we must subtract the probability of getting a matching pair, giving us
   $P\{\text{one left, one right, not matching}\} = \dfrac{12}{25}$ as before.

   (b) **Two more** shoes are picked at random from the barrel without replacing the first two shoes picked back in the barrel. What is the probability that these two shoes are a **matching pair** of shoes?
   **Solution:** If the first two shoes are a matching pair (probability $\frac{1}{25}$), then there are 12 pairs left in the barrel, and the conditional probability that the second set of two shoes is a matching pair is $\frac{1}{23}$ by the same analysis as in part (a). If the first two shoes are not a matching pair (probability $\frac{24}{25}$), then there are 11 pairs of shoes (plus 2 unpaired shoes) in the barrel, and the conditional probability of getting a matching pair is $\frac{11}{\binom{24}{2}} = \frac{22}{24\times 23}$ by the same analysis as in part (a). Hence, the theorem of total probability gives
   $P\{\text{second set is matching}\} = \dfrac{1}{23} \times \dfrac{1}{25} + \dfrac{22}{24\times 23} \times \dfrac{24}{25} = \dfrac{1}{23} \times \dfrac{1}{25} + \dfrac{22}{23} \times \dfrac{1}{25} = \dfrac{1}{25}$.
   This is the *same* as the probability that the first pair is matching! To see this directly, imagine the 26 shoes arranged in a row with all 26! orderings being equally likely. The probability that the first two shoes are a pair is $\frac{1}{25}$. By symmetry, the probability that the third and fourth shoes are a matching pair (or in general, the $(2i-1)$-th and $2i$-th shoes are a matching pair ) is also $\frac{1}{25}$.

(c) What is the probability that there is **at least one matching pair** of shoes in the 4 that have been picked?

**Solution:** There are $\binom{26}{4}$ ways of choosing 4 shoes. If there is *no* matching pair among the 4 chosen, then the four shoes must be one shoe from each of 4 different pairs, and in each pair, there is a choice of which shoe is chosen. Hence,

$P\{\text{no matching pair}\} = \dfrac{\binom{13}{4}\times 2^4}{\binom{26}{4}} = \dfrac{13\times 12\times 11\times 10\times 16}{26\times 25\times 24\times 23} = \dfrac{88}{115}$ after some cancellation and hence $P\{\text{at least one matching pair among four shoes}\} = 1 - \dfrac{88}{115} = \dfrac{27}{115}$.

It is also possible to find the probability of no matching pair assuming that the shoes are drawn in succession using an analysis similar to that of the birthday surprise problem. It does not matter what the first shoe is. The second must not be the mate of the first (24 choices of 25 in the barrel), the third not a mate of the first two (22 choices of 24 in the barrel), and the fourth not a mate of the first three (20 choices of 23 in the barrel).

Hence, $P\{\text{no matching pair}\} = 1 \times \dfrac{24}{25}\times\dfrac{22}{24}\times\dfrac{20}{23} = \dfrac{88}{115}$ after some cancellation.

(d) Given that there are **no matching pairs** of shoes in the **four** that have been picked, what is the (conditional) probability that **all** four shoes are **left** shoes?

**Solution:** The event "four left shoes" occurs if the left shoe is chosen from each of the 4 different pairs in part (c), and hence $P(\text{four left shoes}|\text{no matching pair}) = \dfrac{1}{16}$.

2. Dilbert has **three** coins in his pocket, **two** of which are **fair**, and **one** of which is **biased** with $P(\text{Heads}) = \frac{3}{4}$.

(a) If Dilbert picks **two** coins out of his pocket what is the probability that he did **not** pick the **biased** coin?

**Solution:** There are $\binom{3}{2} = 3$ ways for Dilbert to pick the two coins, and only one of these gives him two fair coins. Hence, $P(\text{did not pick biased coin}) = \frac{1}{3}$.

(b) If Dilbert picks **two** coins out of his pocket, tosses **each** one **once**, and observes a Head and a Tail, what is the (conditional) probability that he did **not** pick the biased coin?

**Solution:** Let $A$ denote the event that Dilbert did not pick the biased coin, i.e., picked the two fair coins, and $B$ the event that the two tosses resulted in one Head and one Tail. We know from part (a) that $P(A) = \frac{1}{3}$. It is also easy to see that $P(B|A) = \frac{1}{2}$ while

$$\begin{aligned} P(B|A^c) &= P(\text{Heads on biased, Tails on fair}) + P(\text{Tails on biased, Heads on fair}) \\ &= \frac{3}{4}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{2} = \frac{1}{2} \end{aligned}$$

Then, $P(A|B) = \dfrac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} = \dfrac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{3}} = \dfrac{1}{3} = P(A)$.

In fact, $A$ and $B$ are *independent events.*

3. Each box of Cornies, the breakfast of silver medalists, contains a picture of either Britney Spears or Paris Hilton. The purchase of each box of Cornies can be regarded as an independent trial of an experiment on which events $S$ and $H$ occur with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

(a) Let $\mathcal{X}$ denote the number of boxes of Cornies purchased till the experimenter has acquired **at least one picture** of each woman. What is $P\{\mathcal{X} = k\}$ for $k \geq 2$? What is $\mathsf{E}[\mathcal{X}]$?

**Solution:** For $k \geq 2$, $\mathcal{X} = k$ if and only if the outcome is either $SSS\cdots SH$ or $HHH\cdots HS$ where a string of $k-1$ $S$'s precedes the $H$ that ends the string or a string of $k-1$ $H$'s precedes the $S$ that terminates the string.

Hence, for $k \geq 2$, $P\{\mathcal{X}=k\} = \left(\frac{1}{4}\right)^{k-1}\frac{3}{4} + \left(\frac{3}{4}\right)^{k-1}\frac{1}{4} = \frac{3+3^{k-1}}{4^k}$.

$$\mathsf{E}[\mathcal{X}] = \sum_{k=2}^{\infty} k \cdot \left(\left(\frac{1}{4}\right)^{k-1}\frac{3}{4} + \left(\frac{3}{4}\right)^{k-1}\frac{1}{4}\right)$$
$$= \frac{3}{4}\left(2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right)^2 + 4\left(\frac{1}{4}\right)^3 + \cdots\right) + \frac{1}{4}\left(2\left(\frac{3}{4}\right) + 3\left(\frac{3}{4}\right)^2 + 4\left(\frac{3}{4}\right)^3 + \cdots\right)$$

But, $1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$ and hence we get

$$\mathsf{E}[\mathcal{X}] = \frac{3}{4}\left(\frac{1}{(1-\frac{1}{4})^2} - 1\right) + \frac{1}{4}\left(\frac{1}{(1-\frac{3}{4})^2} - 1\right) = \frac{4}{3} - \frac{3}{4} + 4 - \frac{1}{4} = 4\frac{1}{3}.$$

Two alternative derivations of this result are of interest. Recall that $\mathsf{E}[\mathcal{X}] = \sum_{i=0}^{\infty} P\{\mathcal{X} > i\}$.
Now, $P\{\mathcal{X} > 0\} = 1$ while for $i \geq 1$, the event $\{\mathcal{X} > i\}$ occurs if and only if the first $i$ trials all resulted in pictures of the same woman.

That is, $P\{\mathcal{X} > i\} = P\{SS\cdots S\} + P\{HH\cdots H\} = \left(\frac{1}{4}\right)^i + \left(\frac{3}{4}\right)^i$.

Hence, $\mathsf{E}[\mathcal{X}] = 1 + \sum_{i=1}^{\infty}\left(\frac{1}{4}\right)^i + \left(\frac{3}{4}\right)^i = \frac{1}{1-\frac{1}{4}} + \frac{1}{1-\frac{3}{4}} - 1 = 4\frac{1}{3}$.

The third argument involves conditional probabilities and the notion of conditional expectation. If $S$ occurs on the first trial, then on average, the number of additional trials to observe the first occurrence of $H$ is $\frac{1}{3/4} = \frac{4}{3}$, i.e., the total number of trials is $\frac{7}{3}$. On the other hand, if $H$ occurs on the first trial, then on average, the number of additional trials to observe the first occurrence of $S$ is $\frac{1}{1/4} = 4$, i.e., the total number of trials is 5.

Hence, the average number of trials is $\frac{1}{4} \times \frac{7}{3} + \frac{3}{4} \times 5 = \frac{7}{12} + \frac{15}{4} = \frac{7+45}{12} = 4\frac{1}{3}$.

(b) Let $\mathcal{Y}$ denote the number of boxes of Cornies purchased till the experimenter has acquired **at least two pictures** of each woman. What is $P\{\mathcal{Y}=4\}$?

**Solution:** $\mathcal{Y} = 4$ if the outcome is one of $SSHH, SHSH, SHHS, HSSH, HSHS, HHSS$ each of which has probability $\left(\frac{1}{4} \times \frac{3}{4}\right)^2 = \frac{9}{2^8}$. Hence, $P\{\mathcal{Y}=4\} = 6 \times \frac{9}{2^8} = \frac{27}{128}$.

4. (a) If $\mathcal{X}$ is a **binomial** random variable with parameters $(4, \frac{1}{3})$, what are the **mean** and **variance** of the random variable $2 + 3\mathcal{X}$?

**Solution:** $\mathsf{E}[\mathcal{X}] = 4 \times \frac{1}{3} = \frac{4}{3}$; $\mathsf{var}(\mathcal{X}) = 4 \times \frac{1}{3} \times \frac{2}{3} = \frac{8}{9}$.

$\mathsf{E}[2+3\mathcal{X}] = 2 + 3 \cdot \mathsf{E}[\mathcal{X}] = 2 + 3 \times \frac{4}{3} = 6.$ $\quad\mathsf{var}(2+3\mathcal{X}) = 3^2 \cdot \mathsf{var}(\mathcal{X}) = 9 \times \frac{8}{9} = 8.$

(b) Let $\mathcal{Y}$ be a **geometric** random variable with parameter $p$ where the value of $p$ is unknown. It is observed that $\{\mathcal{Y} = k\}$. What is the **maximum-likelihood estimate** $\hat{p}_{\text{ML}}$ of the parameter $p$?

**Solution:** The likelihood function is $f(p) = p(1-p)^{k-1}$ which is positive on $(0,1)$ and satisfies $f(0) = f(1) = 0$. Its derivative $(1-p)^{k-1} - p(k-1)(1-p)^{k-2}$ is 0 at $p$ satisfying $(1-p) - p(k-1) = 0$, i.e., at $p = \frac{1}{k}$. Hence, $f(p)$ attains a maximum at $p = \frac{1}{k}$ and so $\hat{p}_{\text{ML}} = \frac{1}{k}$. **Exercise:** Why can we be sure without checking the second derivative that $\frac{1}{k}$ a maximum, and not a minimum, of $f(p)$?