University of Illinois Fall 2006

ECE 413: Final Examination 3 hours

1. A, B, and C are independent events of probabilities 0.3, 0.4 and 0.5 respectively. Find $P(A \cup (B \cap C))$ and $P(A \cup B | B \oplus C)$.

2. $\mathcal{X}$ denotes a Poisson random variable with parameter $\ln(3)$. Find the numerical values of the mean and variance of $\mathcal{Y} = \cos(\pi \mathcal{X})$.

3. $\mathcal{X}$ denotes a continuous random variable with pdf $f_{\mathcal{X}}(u)$ satisfying $f_{\mathcal{X}}(u) = f_{\mathcal{X}}(-u)$ for all $u, \ -\infty < u < \infty$. Suppose that $\mathsf{var}(\mathcal{X}) = 9$. Let $\mathcal{Y} = |\mathcal{X}|$ and $\mathcal{Z} = -\mathcal{X}$, and consider the statements below for all random variables satisfying these conditions.
Mark ALWAYS if the statement is *true for all* such random variables; mark NEVER if the statement is *false for all* such random variables; and mark MAYBE if the statement is *true for some* such random variables *but not all* such random variables.

ALWAYS	NEVER	MAYBE	
☐	☐	☐	$P\{\mathcal{X} > \alpha\} = F_{\mathcal{X}}(-\alpha)$ for all $\alpha, \ -\infty < \alpha < \infty$.
☐	☐	☐	$F_{\mathcal{Y}}(v) = 2F_{\mathcal{X}}(v) - 1$ for $v \geq 0$, and 0 for $v < 0$.
☐	☐	☐	$F_{\mathcal{Z}}(w) = F_{\mathcal{X}}(-w)$ for all $w, \ -\infty < w < \infty$.
☐	☐	☐	$f_{\mathcal{Z}}(w) = f_{\mathcal{X}}(-w)$ for all $w, \ -\infty < w < \infty$.
☐	☐	☐	$\mathsf{E}[\mathcal{Y}^2] = 9$.
☐	☐	☐	$\mathsf{E}[\mathcal{Y}] = 3$.
☐	☐	☐	$\mathsf{var}(\mathcal{Y}) < 9$.
☐	☐	☐	$\mathsf{var}(\mathcal{Y}) < 8$.
☐	☐	☐	$\mathsf{E}[\mathcal{X}\mathcal{Y}] = 0$.
☐	☐	☐	$\mathcal{X}$ and $\mathcal{Y}$ are uncorrelated random variables.
☐	☐	☐	$\mathcal{X}$ and $\mathcal{Y}$ are independent random variables.
☐	☐	☐	$F_{\mathcal{X}}(6) \geq \frac{7}{8} = 0.8750$.
☐	☐	☐	$F_{\mathcal{X}}(6) \geq 0.9772$.

4. A radio-frequency signal is either a radar echo (hypothesis H_1) or ambient noise (hypothesis H_0). The *phase* of the signal is modeled as a continuous random variable $\mathcal{X}$ whose pdf is as follows:

- When H_0 is true, $\mathcal{X}$ has pdf $f_0(u) = \begin{cases} \frac{1}{2\pi}, & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$
- When H_1 is true, $\mathcal{X}$ has pdf $f_1(u) = \begin{cases} \frac{1}{2\pi}\left(1 + \cos u\right), & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$

The radar receiver measures $\mathcal{X}$ and decides which hypothesis is true.

(a) Suppose that the *maximum-likelihood* decision rule is being used. What value(s) of $\mathcal{X}$ result in a decision in favor of H_1?

(b) Find the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

(c) Now suppose that $P(\mathsf{H}_0) = \pi_0 = \frac{1}{3}, \ P(\mathsf{H}_1) = \pi_1 = \frac{2}{3}$. What is the *average* error probability $\bar{P}_e$ of the maximum *a posteriori* probability (MAP) (that is, minimum-error-probability or Bayesian) decision rule?

(d) For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_0 regardless of the value of $\mathcal{X}$?

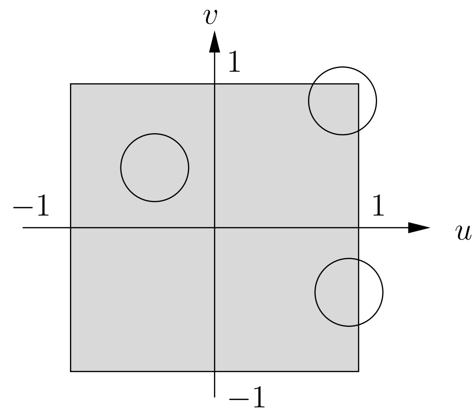
5. A professor breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.1$ per minute.

(a) What is the expected length of time between two successive chalk breaks?

(b) What is the average number of times that the professor breaks the chalk during a 50 minute lecture?

(c) *Given* that the professor broke 6 chalk pieces in 50 minutes, what is the average number of pieces he broke in the first 25 minutes?

6. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the interior of a square of side 2 centered at the origin. Consider a circle of radius $r < 1$ centered at $(\mathcal{X}, \mathcal{Y})$, and let $\mathcal{Z} \in \{0, 1, 2\}$ denote the number of sides of the square that are crossed by the circle, as illustrated in the figure below.



If $\mathsf{E}[\mathcal{Z}] = \dfrac{3}{2}$, what is the value of r?

7. The joint pdf of $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(**a**) Are $\mathcal{X}$ and $\mathcal{Y}$ independent random variables? SHOW YOUR WORK.

(**b**) Find the pdf of $\mathcal{Z} = \mathcal{X}\mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.

8. The jointly Gaussian random variables $\mathcal{X}$ and $\mathcal{Y}$ have means 0 and 14 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{16}$.

(**a**) Find the pdf of the random variable $\mathcal{Z} = 5\mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.

(**b**) Find the numerical value of $P\{\mathcal{Y} > 3\mathcal{X}\}$.