University of Illinois

ECE 313 **FALL 2007**

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TEST I

7:00 p.m. - 8:30 p.m.

NOTE: This is a **closed-book closed-notes** (and closed-neighbor) exam, with only one sheet of notes ($8\frac{1}{2}'' \times 11''$) allowed. Also, no calculators, laptops, palm pilots, and the like are allowed.

Name (Last, First) : ....

Section : ◯ Section C, 10 MWF ◯ Section D, 11 MWF

Problem 1 : .... ($2 \times 10 = 20$ points)

Problem 2 : .... ($4 \times 10 = 40$ points)

Problem 3 : .... ($14 + 6 = 20$ points)

Problem 4 : .... ($6 + 7 + 7 = 20$ points)

TOTAL : .... (100 points)

Problem 1 (20 points)

Let E, F, and G be three events defined on a common sample space, with $0 < P(E) < 1$, $0 < P(F) < 1$, and $0 < P(G) < 1$. (*Note the strict inequality in each case.*)

Read each of the following statements carefully, and check the corresponding True box if the statement is always true (that is, it holds for all E, F, G); otherwise check the corresponding False box. Each correct choice counts +2 points, whereas an incorrect choice counts −1 point; so guess at your own risk.
You can use the space provided at the bottom as well as back of the previous page for scratch work.

TRUE	FALSE	
☐	☐	$P(EF) + P(E^c \cup F^c) = 1$
☐	☐	$P(E \cup F) = 1 - P(E^c \| F^c)P(F^c)$
☐	☐	$P(E \| F^c) + P(E \| F) = 1$
☐	☐	$P(EF \| E) = P(EF \| F)$
☐	☐	$P(E \| F) = P(E \| FG)P(G \| F) + P(E \| FG^c)P(G^c \| F)$
☐	☐	$P(EFG) \leq \min \{P(E), P(F), P(G)\}$
☐	☐	$P(E \| F)P(F) + P(E^c \| F)P(F) = P(F)$
☐	☐	If $P(EFG) = P(E)P(F)P(G)$, then E, F, G are independent
☐	☐	If E and F are mutually exclusive, $P(E \| F) = P(E)$
☐	☐	If the occurrence of event F makes event E more likely, then the occurrence of E necessarily makes F also more likely.

END OF PROBLEM 1

Problem 2 (40 points)

In each question given below and on the next page, only one of the given four answers is correct. In each case, **circle** the *letter* corresponding to the correct answer. Here each correct choice counts +4 points, whereas an incorrect choice receives −1 point. There is a total of 10 questions, 5 on this page and another 5 on the next. You can use the space provided at the bottom of each page as well as the facing pages for scratch work.

(i) Let E, F, G be three events defined on a common sample space, with the properties: E and G are mutually exclusive, and $P(EF) = P(FG) = 0.2$, $P(E) = P(G) = 0.4$, $P(E \cup F \cup G) = 1$. Compute $P(F)$.

a. 0.4 **b.** 0.6 **c.** 0.2 **d.** none of these

(ii) In (i) above, $P(F|E)$, the conditional probability of F given E, is

a. 0.6 **b.** 0.5 **c.** 0.4 **d.** none of these

(iii) Let X be a binomial random variable with parameters $(n, p) = (10, 0.4)$. Variance of X is

a. 2.4 **b.** 1.6 **c.** 4 **d.** 6

(iv) Let X be a Poisson random variable with parameter $\lambda = 3$. $P(X \geq 2)$, the probability that X is larger than or equal to 2, is

a. $4e^{-3}$ **b.** $(17/2)e^{-3}$ **c.** $1 - 4e^{-3}$ **d.** none of these

(v) Let E and F be two independent events, with $P(E) = P(F) = 0.25$. Let G be another event on the same sample space, which is disjoint from both E and F, and with $P(G) = 0.25$. Then, $P(G|F)$ is

a. 1 **b.** 0.5 **c.** 0.25 **d.** 0

(vi) Let X be a discrete random variable with probability mass function (pmf)

$$P(X = i) = \begin{cases} 0.4 & \text{for } i = -1 \\ 0.2 & \text{for } i = 1 \\ 0.4 & \text{for } i = 2 \end{cases}$$

The expected value of X, $E[X]$, is

a. 0.6 **b.** 1 **c.** 1.6 **d.** 2.2

(vii) Let X be as defined in (vii) above. $E[(X^2 - 1)^2]$ is

a. 1.2 **b.** 3.6 **c.** 2 **d.** none of these

(viii) Y is a discrete random variable with mean 3 and variance 4. Use Chebyshev's inequality to obtain the value of K such that $P(|Y - 3| > 4) \leq K$.

a. $K = 1$ **b.** $K = 0.5$ **c.** $K = 2$ **d.** $K = 0.25$

(ix) You have two biased coins. The probability of heads for the first one is equal to 0.6 and for the second one equal to p, where p is unknown. Consider the experiment where you flip the coins together a total of 500 times (independent trials), and record the number of times both show heads. Let that number be 120. What is the maximum likelihood estimate, $\hat{p}$, for p?

a. 0.36 **b.** 0.4 **c.** 0.5 **d.** none of these

(x) In the experiment in (ix) above, you now replace the second coin with an unbiased one. Let X be the number of times both coins show heads when flipped together 500 times. What is the most likely value for X? That is, find the integer k that maximizes the quantity $P(X = k)$.

a. 150 **b.** 151 **c.** 120 **d.** none of these

END OF PROBLEM 2

Problem 3 (14+6 = 20 points)

The probability of a bit error in a communication line is 10^{-3}. We want to compute the probability that a block of $2,000$ bits has 5 or more errors.

(i) Compute the desired error probability when bit error is modeled as a Poisson random variable. [You can leave the final answer in terms of exponentials.]

Probability when modeled as Poisson =

(ii) Now consider modeling the bit error as a Binomial random variable.

(a) Write down the expression for the desired error probability under this modeling assumption.

(b) Can you use the result in part (i) above to compute this quantity? Why?

(a) Expression for probability when modeled as Binomial =

(b)

END OF PROBLEM 3

Problem 4 (6+7+7 = 20 points)

Consider a channel with input X, a Bernoulli random variable, and output Y, also a Bernoulli random variable, with conditional probabilities: $P(Y = 1|X = 0) = \alpha\,,\quad P(Y = 0|X = 1) = \beta\,,$ where α and β are two positive parameters, each one less than 1. Such a channel is called a *binary channel*, which transmits 0's and 1's, but with some probability converts 0's to 1's and likewise 1's to 0's, and hence the receiving end does not know exactly whether what was sent was 0 or 1. Based on the observation of the output of the channel, Y, we would like to determine the input, X. This problem pertains to the three different hypothesis testing schemes discussed in class for this purpose.

(i) First suppose that the maximum likelihood (ML) decision rule is used. What is the probability of error if in fact $X = 0$ was sent? Do the same for the case when in fact $X = 1$ was sent.
[You should express your answers in terms of the parameters α and β.]

$P(\text{error}|X = 0) =$

$P(\text{error}|X = 1) =$

(ii) We now let $\alpha = 0.1$ and $\beta = 0.3$, and further take $P(X = 0) = p$, where p is a parameter, $0 < p < 1$. Find the Maximum A Posteriori Probability (MAP) decision rule and the corresponding probability of error.
[You should express the answers in terms of the parameter p, by considering all possible values.]

MAP detection rule =

$P(\text{error}) =$

(iii) Lastly, we introduce a cost function $C_{00} = -1, C_{01} = 3, C_{10} = 6, C_{11} = -3$, where C_{ij} is the cost of ruling that $X = i$ when in fact it is $X = j$, $\quad i, j = 0, 1$. Let α and β be as in part (ii) above, and $p = 0.25$. Find the Bayes' minimum average cost decision rule.

Bayes' minimum average cost DR =

END OF PROBLEM 4 / END OF TEST

SCRATCH SHEET