University of Illinois Fall 2006

ECE 413: Problem Set 12

Due:	Wednesday November 29 at the beginning of class.
Reading:	Ross, Chapter 6
Noncredit exercises:	Ross, Chapter 6, Problems 1, 9-15, 20-23

This Problem Set contains six problems

1. A signal $x(t) = \exp(-\pi t^2)$, $-\infty < t < \infty$, is the input to an ideal low-pass filter whose transfer function is $H(f) = \text{rect}(f/2)$. Let $y(t)$ denote the output of the filter. Find the *numerical* value of $y(0)$. [Hint: $X(f) = \exp(-\pi f^2)$, $-\infty < f < \infty$.]

2. Consider the TMR system discussed in Problem 3 of Problem Set 10 with $\lambda = -\ln 0.999$.

   (a) Sketch the reliability functions $R(t) = P\{\mathcal{X}_1 > t\}$ and $R_{\text{TMR}}(t) = P\{\mathcal{Y} > t\}$ of a single module and the TMR system respectively.

   (b) Find and sketch the hazard rates $h(t)$ and $h_{\text{TMR}}(t)$of a single module and the TMR system respectively.

3. If hypothesis H_0 is true, the pdf of $\mathcal{X}$ is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is exponential with parameter 10.

   (a) Sketch the two pdfs.

   (b) State the *maximum-likelihood* decision rule in terms of a threshold test on the *observed value* u of the random variable $\mathcal{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ to 1.

   (c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part(b)?

   (d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_0/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathcal{X}$.

   (e) If $\pi_0 = 1/3$, what is the *average* probability of error of the Bayesian decision rule?

   (f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by $\mathcal{X}$?

   (g) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathcal{X}$. What is the average probability of error for the maximum-likelihood rule when $\pi_0 > 2/3$?

4. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 0.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 0 \le u+v < 1, \\ 1.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 1 \le u+v < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the marginal pdf of $\mathcal{X}$.

   (b) Find $P\{\mathcal{X} + \mathcal{Y} \le 3/2\}$ and $P\{\mathcal{X}^2 + \mathcal{Y}^2 \ge 1\}$.

5. Ross, Problem 8, page 313.

6. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathcal{X},\mathcal{Y}}(u,v)$ is nonzero.

(b) Find the marginal pdfs of $\mathcal{X}$ and $\mathcal{Y}$.

(c) Are the random variables $\mathcal{X}$ and $\mathcal{Y}$ independent ?

(d) Find $P\{\mathcal{Y} > 3\mathcal{X}\}$.

(e) For $\alpha > 0$, find $P\{\mathcal{X} + \mathcal{Y} \leq \alpha\}$.

(f) Use the result in part (e) to determine the pdf of the random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$.