University of Illinois Fall 2006

ECE 413: Problem Set 10

Due: Wednesday November 8 at the beginning of class.
Reading: Ross, Chapter 5

This Problem Set contains six problems

1. A mailman starting a new route estimates that the probability that he is bitten by a dog on the very first day is $\frac{1}{2}$. As each day passes without the mailman being bitten, he grows increasingly wary (and he becomes more cognisant of the locations of the dogs on his route). The *conditional* probability that the mailman is bitten on the n-th day, given that he has not been bitten on days 1 through $n-1$, is $\frac{1}{n+1}$.

   Let $\mathcal{X}$ denote the day on which the mailman is first bitten by a dog (at which time he goes postal, shoots the dog, and is transferred to a new route.)

   (a) What is the pmf of $\mathcal{X}$? Hint: first find $P\{\mathcal{X} > n\}$ for $n = 1, 2, \ldots$

   (b) (This one is for dog lovers everywhere.) Find the expected value of $\mathcal{X}$ from the pmf that you found in part (a).

   (c) According to Problem 4(a) of Problem Set 9, $\mathsf{E}[\mathcal{X}] = \sum_{n\geq 0} P\{\mathcal{X} > n\}$. Use this result to find $\mathsf{E}[\mathcal{X}]$ from the $P\{\mathcal{X} > n\}$ values that you found in part (a).

   (d) Let $\mathcal{Y}$ denote a random variable obtained from $\mathcal{X}$ as follows. We toss a fair coin, and set $\mathcal{Y} = \mathcal{X}$ or $\mathcal{Y} = -\mathcal{X}$ according as the coin turns up Heads or Tails. What is the pmf of $\mathcal{Y}$? What is the expected value of $\mathcal{Y}$? Explain your answer.

2. $\mathcal{X}$ is a *uniform* random variable with $\mathsf{E}[\mathcal{X}] = 1$ and $\mathsf{var}(\mathcal{X}) = 3$. Find $P\{\mathcal{X} < 0\}$ and $\mathsf{E}[|\mathcal{X}|]$.

3. As discussed in class, the probability of failure of a TMR system with (perfect majority gate) is $3p^2 - 2p^3$ where p is the probability of failure of each module, and the modules fail independently of each other. Now, suppose that the system is put into operation at $t = 0$, and let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ denote the time of failure of each module. The independence of failures enters into our calculations as the assertion that for all $t_1, t_2, t_3 > 0$, the events $\{\mathcal{X}_1 > t_1\}, \{\mathcal{X}_2 > t_2\}, \{\mathcal{X}_1 > t_3\}$ are independent events. Note that the occurrences of these events are equivalent to the assertions that modules 1, 2, 3 respectively have *not* failed (i.e., are operational) at times t_1, t_2, t_3. We model $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ as exponential random variables with parameter λ.

   Let $\mathcal{Y}$ denote the time of failure of the TMR system so that the occurrence of the event $\{\mathcal{Y} > T\}$ means that the TMR system is operational at time T.

   (a) Express the event $\{Y > T\}$ in terms of unions, intersections and complements of the events $\{\mathcal{X}_1 > T\}, \{\mathcal{X}_2 > T\}, \{\mathcal{X}_3 > T\}$.

   (b) Show that $P\{\mathcal{Y} > T\} = 3\exp(-2\lambda T) - 2\exp(-3\lambda T)$, and use this result to find $\mathsf{E}[\mathcal{Y}]$, the *average lifetime* of the TMR system. [Hint: $\mathsf{E}[\mathcal{Y}] = \int_0^\infty P\{\mathcal{Y} > T\}\, dT$.] The average lifetime is also known as the mean time before failure (MTBF) or mean time to failure (MTTF) in the reliability literature.

   (c) Find the *median* value of $\mathcal{Y}$ by solving the equation $P\{\mathcal{Y} > T\} = \frac{1}{2}$ for T.

   (d) Compare your answers of parts (b) and (c) to the MTBF λ^{-1} and the median lifetime $\lambda^{-1}\ln 2$ for a single module. Do the answers surprise you? Is the TMR system a more reliable system as claimed?

   (e) Now suppose that $\lambda = -\ln 0.999$. What are the numerical values of $P\{\mathcal{X}_1 > 1\}$ and $P\{\mathcal{Y} > 1\}$?

(f) I hope you found in part (e) that $P\{\mathcal{X}_1 > 1\} = 0.999$ and so a single module works with 99.9% reliability for at least one unit of time. What is the largest value of T for which $P\{\mathcal{Y} > T\} \geq 0.999$? How does the TMR system compare to a single module in terms of providing 99.9% reliability over long periods of time?

4. Let $\mathcal{X}$ denote a unit Gaussian random variable. Its CDF is $\Phi(u)$.

   (a) What is the derivative of $\exp(-u^2/2)$? Use this result to compute $\mathsf{E}[|\mathcal{X}|]$.

   (b) $Q(x) = \displaystyle\int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{u^2}{2}\right) du = P\{\mathcal{X} > x\}$ is called the *complementary* CDF. A useful bound is $Q(x) \leq \frac{1}{2}\exp(-x^2/2)$ for $x \geq 0$. Derive this bound by first proving that $t^2 - x^2 > (t-x)^2$ for $t > x > 0$ and then applying this to

$$\exp(x^2/2)Q(x) = \int_x^\infty (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{t^2 - x^2}{2}\right) dt.$$

5. Let $\mathcal{X}$ denote a Gaussian random variable with mean -10 and variance $\sigma^2 = 4$. You have at your disposal two calculators: one can calculate $\Phi(x)$ for $x \geq 0$, and the other can calculate $Q(x)$ for $x \geq 0$. Both have the usual assortment of basic arithmetic functions. Write down expressions for calculating each of the following probabilities with each of the calculators. Remember that the argument of Φ or Q must be ≥ 0 in all cases.
   (**a**) $P\{\mathcal{X} < 0\}$. (**b**) $P\{-10 < \mathcal{X} < 5\}$. (**c**) $P\{|\mathcal{X}| \geq 5\}$. (**d**) $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\}$.

6. The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

   (a) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

   (b) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?