University of Illinois Fall 2006

ECE 413: Problem Set 8

Due: Wednesday October 25 at the beginning of class.
Reading: Ross, Chapters 3, 4, 5, and the notes on decision-making on the class web page
Reminder: No class on Friday October 27
Cancelled class will be made up on Monday October 30, 7-8 pm

This Problem Set contains five problems

1. The number of α-particles emitted by a source during a unit time interval can be modeled as a Poisson random variable $\mathcal{X}$ with parameter λ. The α-particles are detected by an (imperfect) Geiger counter which detects each particle with probability $p < 1$. Let $\mathcal{Y}$ denote the number of particles detected by the Geiger counter. Each detection can be considered to be an independent event, and so the *conditional* pmf of $\mathcal{Y}$ *given* that $\mathcal{X} = n$ is a binomial pmf with parameters (n, p). Thus,

$$p_{\mathcal{Y}|\mathcal{X}=n}(k|\mathcal{X} = n) = P\{\mathcal{Y} = k|\mathcal{X} = n\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } 0 \le k \le n.$$

   (a) What is the conditional mean of $\mathcal{Y}$ given the event $\{\mathcal{X} = n\}$?
   (b) From the result of part (a), find the *unconditional* mean of $\mathcal{Y}$.
   (c) What is the unconditional pmf of $\mathcal{Y}$? [Hint: if $\mathcal{Y} = k$, then $\mathcal{X}$ must have taken on some value $\ge k$; $\mathcal{X}$ cannot possibly be smaller than k.]
   (d) What is the conditional pmf of $\mathcal{X}$ given that $\mathcal{Y} = k$?
   (e) What is the conditional mean of $\mathcal{X}$ given that $\mathcal{Y} = k$?
   (f) We can observe the Geiger counter reading $\mathcal{Y}$, but we wish to know the value of $\mathcal{X}$. If the Geiger counter reading is k, i.e. the event $\{\mathcal{Y} = k\}$ is observed, what is the maximum likelihood estimate of $\mathcal{X}$?

2. The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. All members of the group then vote in accordance with the caucus decision. If you believe that this is the way politics works, I have this beautiful skyscraper on Wacker Drive in Chicago that I am willing to sell to you at a real bargain price . . .

   (a) Let A, B, C, and D respectively denote the events that the four groups vote to eliminate all income taxes on capital gains. Suppose that the probabilities of these independent events are $P(A) = 0.9, P(B) = 0.6, P(C) = 0.5$ and $P(D) = 0.2$. What is the probability that the bill passes?
   (b) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities $P(E) = 0.99, P(F) = 0.4, P(G) = 0.6$, and $P(H) = 0.1$, what is the probability that the motion to override the veto passes ?

   Political innocents are reminded that a simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

3. The dice game of *craps* begins with the player (called the *shooter*) rolling two fair dice. If the sum is 2, 3, or 12, the shooter loses the game. If the sum is a 7 or 11, the shooter wins the game. If the sum is any of 4, 5, 6, 8, 9, 10, then the shooter has neither won nor lost (as yet). The number rolled is called the shooter's *point*, and what happens next is described in parts (b) and (c) below.

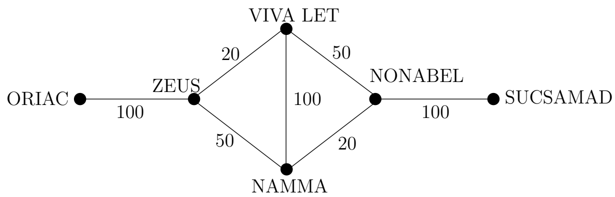
(a) What is the probability that the shooter wins the game on the first roll? What is the probability that the shooter loses the game on the first roll? What is the probability that the shooter's point is $i, i \in \{4, 5, 6, 8, 9, 10\}$? I need six answers here, folks!

(b) Suppose that the shooter's point is i. The shooter rolls the dice again. If the result is i, the shooter is said to have *made the point* and wins the game. If the result is 7, the shooter loses the game (craps out). If the result is anything else, the shooter rolls the dice again. This continues until the shooter either makes the point or craps out. For each $i \in \{4, 5, 6, 8, 9, 10\}$, compute the probability that the shooter wins the game. Note that these are *conditional* probabilities of winning given that the shooter's point is i.

(c) Conditioned on the shooter's point being i, what is the expected number of dice rolls till the game ends? (Note: one dice roll = rolling two dice simultaneously). What is the expected number of dice rolls in a game of craps? What is the (unconditional) probability of winning at craps?

(d) If the shooter's point is 8, then side-bets are offered at 10 to 1 odds that the shooter will make the point *the hard way* by rolling $(4, 4)$. Is this a fair bet? (Remember that 10 to 1 odds means if you bet a dollar, you will either lose the dollar, or you will win ten dollars (and will also get your original dollar back, of course!)).

4. MiddleEast Bell, a wholly owned subsidiary of Psingular Corp. has built a telephone network as shown below. Each link fails with probability p and link failures are independent events.



If a link fails, switches automatically re-route calls so as to avoid the failed link (if possible).

(a) What is the probability of being able to call from ORIAC to SUCSAMAD?

(b) Given that it is possible to call from ORIAC to SUCSAMAD, what is the conditional probability that the ZEUS to NAMMA link is in working condition?

(c) The link capacities (i.e., the numbers of telephone calls that the links can carry (in either direction)) are as marked on the diagram. Let $\mathcal{X}$ denote the number of calls that can be made from ORIAC to SUCSAMAD. Find the pmf and the expected value of $\mathcal{X}$

5. The ToyAuto Company needs to decide which of the following two methods provides more reliable transportation:

- a single gigantic car with N engines, N transmissions, N brakes, ... etc. that works (i.e. provides us with transportation) as long as *at least one* of its engines and *at least one* of its transmissions, and *at least one* of its brakes ... works.
- N separate ordinary cars that fail as soon as any one of their parts fail, but which together provide us with transportation as long as at least one car is in working condition.

Each car is made of M different types of parts, and (at least) one part of each different type must work for the car to work. Each part fails with probability p and all the failures are independent events.

(a) For each method, find the probability of system failure (we have no transportation!) in terms of p, N and M

(b) Suppose that $M = 5$ and $p = 0.2$. If it is desired that the system failure probability be less than 0.001, what should N be with each method?

(c) Repeat part (b) assuming that $M = 1000$.