University of Illinois Fall 2006

ECE 413: Problem Set 7

Due: Wednesday October 18 at the beginning of class.
Reading: Ross, Chapters 3, 4, 5, and the notes on decision-making on the class web page

This Problem Set contains five problems

1. ["From the town of Bedrock, They're a page right out of history ..."] In Problem 3(a) of Hour Exam I, Fred and Wilma were playing a series of games by taking turns tossing a coin with $P(\text{Heads}) = p$ and $P(\text{Tails}) = q = 1 - p$. Fred tossed first in the first game, then Wilma, then Fred again, and so on until the first game ended. The *loser* of a game tossed first in the next game. Let F_n and W_n respectively denote the events that Fred and Wilma won the n-th game. For the purposes of *this* problem, assume that the coin is a fair coin, and thus $P(F_1) = \frac{2}{3}, P(W_1) = \frac{1}{3}, P(F_2) = \frac{4}{9}, P(W_2) = \frac{5}{9}$.

   (a) Is it true that Fred and Wilma *always* alternated in tossing the coin regardless of whether a coin toss ended a game or not?

   (b) Express $P(F_{n+1})$ and $P(W_{n+1})$ in terms of $P(F_n)$ and $P(W_n)$.

   (c) Your answers to part (b) are a pair of *difference* equations for $P(F_n)$ and $P(W_n)$. Show that the solutions are

   $$P(F_n) = \frac{1}{2}\left[1 - \left(\frac{-1}{3}\right)^n\right] \quad \text{and} \quad P(W_n) = \frac{1}{2}\left[1 + \left(\frac{-1}{3}\right)^n\right]$$

   [Hint If you never learned how to solve difference equations in Math 486 or ECE 410, assume that for *each integer* $n \geq 1$, it is possible to express $P(F_n)$ as $a + b\alpha^n$. Substitute into the difference equation and use the fact that equality must hold for *all* n to find a and α. The value of b is obtained from the initial condition $P(F_1) = \frac{2}{3}$. The solution for $P(W_n)$ is similar but the initial condition is $P(W_1) = \frac{1}{3}$.]

   (d) Show that $\lim_{n\to\infty} P(F_n) = \lim_{n\to\infty} P(W_n) = \dfrac{1}{2}$ so that if Fred and Wilma are still playing, the game is quite close to being fair by now!

2. ["I'm leaving on a jet plane ..."] Consider again Problem 8 of Problem Set 5 in which 15 of the 105 passengers who hold reservations are arriving in Chicago on a connecting flight. If the connecting flight is on time, all 15 show up for the flight to Champaign; else, none of the 15 shows up. Let $\mathcal{Y}$ denote the number of nonconnecting passengers who actually show up for the flight. Let H_0 denote the hypothesis that the connecting flight is late, and H_1 the hypothesis that the connecting flight is on time. It is reasonable to assume that the pmf of $\mathcal{Y}$ is the same regardless of which hypothesis is true, and hence we model $\mathcal{Y}$ as a binomial random variable with parameters (90, 0.9). On the other hand, $\mathcal{X}$, the total number of passengers showing up for the flight, equals $\mathcal{Y}$ if H_0 is true, while if H_1 is true, then $\mathcal{X} = 15 + \mathcal{Y}$, and thus the pmf of $\mathcal{X}$ *does* depend on which hypothesis is true.

   (a) Suppose that the gate agent observes that $\mathcal{X} = 86$. What is $P\{\mathcal{X} = 86\}$ when H_0 is the true hypothesis? What is $P\{\mathcal{X} = 86\}$ when H_1 is the true hypothesis? What is the value of the likelihood ratio when $\mathcal{X} = 86$? What is the agent's maximum-likelihood decision as to whether the connecting flight is late?

   (b) Repeat part (a) for the case when the gate agent observes that $\mathcal{X} = 96$.

   (c) The gate agent knows that $\pi_0 = P\{\mathsf{H}_0 \text{ is the true hypothesis}\} = \frac{7}{8}$. For each of the two observations considered in parts (a) and (b), what is the agent's MAP (or Bayesian or minimum-probability-of-error) decision as to whether the connecting flight is late?

3. Consider the matrix of Problem 6 of Problem Set 5 as a likelihood matrix. The three hypotheses are that the transmitted signal $\mathcal{X}$ took on values 1, 2, or 3 and the receiver observes that $\mathcal{Y}$ took on values 1, 2, or 3.

   (a) Having observed $\mathcal{Y}$, what is the receiver's maximum-likelihood decision rule as to which signal was transmitted ?

   (b) The receiver knows the pmf of $\mathcal{X}$. What is the receiver's maximum *a posteriori* probability (MAP) decision rule?

4. We return to the baseball pitcher of Problem 2 of Problem Set 6. A fan sitting in the bleachers observes that the batter got a hit (the event H), but is too far away to be able to tell what kind of pitch it was.

   (a) What is his *maximum-likelihood* decision rule as to whether the pitch was a fast ball, curve ball or slider?

   (b) Now suppose that after cheering the hit, the fan returns to his seat and finds $P(H) = 0.25$ listed in the program guide. Having lasted in ECE 413 through Problem Set 6, he can compute $P(F), P(C)$ and $P(S)$. But, since he dropped the course immediately after Hour Exam I, please help him compute his maximum *a posteriori* probability decision rule as to what kind of pitch it was.

5. ["Give me an F!" shouted the cheerleader...] $\mathsf{H}_0, \mathsf{H}_1$, and H_2 respectively denote the hypotheses that a student is excellent, good, or average (there are no poor students). The number of grade points earned by the student in a course is a random variable $\mathcal{X}$ that takes on values 3, 6, 9, and 12 only. The professor knows that the pmf of $\mathcal{X}$ when H_0 is true is $p_0(12) = 0.75, p_0(9) = 0.15, p_0(6) = 0.08, p_0(3) = 0.02$, that is, an excellent student has 75% chance of doing well enough on the exam to get an A, 15% chance of a B, etc. Similarly, when H_1 is the true hypothesis, the pmf of $\mathcal{X}$ is $p_1(12) = 0.15, p_1(9) = 0.6, p_1(6) = 0.15, p_1(3) = 0.1$, while if H_2 is true, $p_2(12) = 0.05, p_2(9) = 0.1, p_2(6) = 0.65, p_2(3) = 0.2$. The professor observes $\mathcal{X}$ and must decide which of the hypotheses $\mathsf{H}_0, \mathsf{H}_1$, and H_0 is true.

   (a) What is the professor's maximum-likelihood decision rule?

   (b) What is the probability that an excellent student is mistakenly labeled as good? What is the probability that an excellent student is mistakenly labeled as average? What is the probability that an average student is classified either as good or as excellent?

   (c) If $P(\mathsf{H}_0) = 0.2, P(\mathsf{H}_1) = 0.55$, and $P(\mathsf{H}_2) = 0.25$, what is the probability that the maximum-likelihood decision rule mis-classifies students?

   (d) What is the Bayes' decision rule corresponding to these probabilities and what is the probability that the Bayes' decision rule mis-classifies students?

   (e) At the Lake Wobegon campus of the University, 95% of students are excellent and 5% are good (and thus they are all above average!) What is Bayes' decision rule in this case? That is, what does the Bayesian professor decide about a student based on the four possible results of the student's exam?