University of Illinois Fall 2006

ECE 413: Problem Set 5

Due:	Wednesday **October 4** at the beginning of class.
Reading:	Ross, Chapters 3 and 4
Reminder:	**No class on Wednesday September 27 and Friday September 29**
Noncredit Exercises:	DO NOT turn these in.
	Chapter 3: Problems 1, 2, 5, 10, 12, 16, 31, 38, 39, 44
	Theoretical Exercises 1, 2, 8; Self-Test Problems 1-10.

This Problem Set contains eight problems

1. There are N multiple-choice questions (with 5 possible answers each) on a certain exam. A student knows the answers to K questions and answers them correctly. On the remaining $N-K$ questions, the student guesses randomly among the 5 choices. The examiner knows N, and can observe the values of $\mathcal{C}$, the number of *c*orrect answers, and $\mathcal{W} = N - \mathcal{C}$, the number of *w*rong answers on the answer sheet. Note that $\mathcal{C}$ can have values K, $K+1$, ..., N. What the examiner is really interested in, though, is *estimating* the value of K from the observed values $\mathcal{C}$ and $\mathcal{W}$.

   (a) Explain why it is reasonable to model $\mathcal{W}$ as a binomial random variable with parameters $(N-K, 0.8)$. What assumptions are you making?

   (b) Suppose that n answers are incorrect, that is, $\mathcal{W} = n$ and $\mathcal{C} = N - n$. What is the *likelihood* of this observation? Hint: your answer will depend on N, n and the unknown parameter K that the examiner is interested in estimating.

   (c) Having observed that $\mathcal{W} = n$, the examiner is sure that K cannot exceed $N-n$, i.e., K can have value $0, 1, 2, \ldots, N-n$ only. Use the method of Proposition 6.1 of Chapter 4 in Ross to show that the likelihood you found in part (b) is maximized at $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$.

   (d) Since $\mathcal{C} = N - n$, a *guessing penalty* is applied by subtracting $\lfloor 0.25n \rfloor$ from $\mathcal{C}$ to get an estimate of K. For $N = 100$ and $K = 90$, compare the *examiner's estimate* $\tilde{K} = N - n - \lfloor 0.25n \rfloor$ and the maximum likelihood estimate $\hat{K}$ for $n = 0, 1, \ldots 10$. Notice that lucky guesses cause the examiner to overestimate K while the unlucky student who guesses wrong on all ten problems has to suffer the further indignity of having the score reduced to something smaller than K.

2. An urn contains 10 red balls and an unknown number x of blue balls. The experiment consists of drawing one ball at random from the urn and noting its color. Consider 100 independent trials of this experiment. Thus, the ball drawn is replaced, and the urn shaken well before the next ball is drawn. It is observed that 25 of the drawings resulted in a red ball and 75 in a blue ball.

   (a) What is the maximum likelihood estimate $\hat{x}$ of the number of blue balls x in the urn?

   (b) Let $\hat{p} = \dfrac{10}{10+\hat{x}}$. With what level of confidence can we say that $p = \dfrac{10}{10+x}$ lies in the interval $[\hat{p} - 0.1, \hat{p} + 0.1]$?

   (c) Find the confidence interval for p whose confidence level is 0.96.

3. ["I am from Iowa; I only work in outer space ..."] Each box of Cornies, the breakfast of silver medalists, contains either a picture of Homer Simpson or a picture of Bart Simpson with probabilities $\frac{2}{3}$ and $\frac{1}{3}$ respectively. The contents of each box may be considered to be independent of the contents of other boxes. Little Jimmy T. Kirk of Cedar Rapids, Iowa, asks his mother to buy boxes of Cornies until he has accumulated at least one picture of both Homer and Bart.

   (a) What is the minimum number of boxes that Mrs Kirk must purchase?

   (b) Let $\mathcal{X}$ denote the number of boxes that Mrs Kirk buys till Jimmy has his heart's desire. What is the pmf of $\mathcal{X}$? What is the expected value of $\mathcal{X}$?

   (c) The following year, pictures of Harold and Kumar replace those of Homer and Bart in boxes of Cornies. Being even more spoiled than before, Jimmy wants to have at least *two* pictures of each. Repeat parts (a) and (b) for these conditions.

4. A long message is divided into L packets of N bits each (including headers, addresses, timestamps, data bits, CRC bits, tail, flags etc.) and transmitted over a channel with bit error probability p. If the CRC detects that a packet is received in error, the packet transmission is repeated. *But,* if a packet has been transmitted 5 times and still has not been received correctly on the fifth try, then it is deemed to be lost and is not transmitted again.

   (a) What is the probability that the CRC indicates no error in a received packet?

   (b) What is the probability that a packet is transmitted successfully (i.e. is not deemed to be lost)?

   (c) Let $\mathcal{X}_i$ denote the number of times that the i-th packet is transmiitted. What is the pmf of $\mathcal{X}_i$? What is $\mathsf{E}[\mathcal{X}_i]$?

   (d) What is the probability that none of the L packets are lost?

5. This problem on conditional probability has three unrelated parts:

   (a) If $P(A|B) = 0.3, P(A^c|B^c) = 0.4$, and $P(B) = 0.7$, find $P(A|B^c), P(A)$, and $P(B|A)$.

   (b) If $P(E) = \frac{1}{4}, P(F|E) = \frac{1}{2}$, and $P(E|F) = \frac{1}{3}$, find $P(F)$.

   (c) If $P(G) = P(H) = \frac{2}{3}$, show that $P(G|H) \geq \frac{1}{2}$.

6. Let $\mathcal{X}$ and $\mathcal{Y}$ denote two discrete random variables taking on values $1, 2, 3$. $\mathcal{X}$ denotes a number that we wish to transmit over a channel using one of the three signals s_1, s_2 and s_3. Let $s_{\mathcal{X}}$ denote the signal that is transmitted. Noise in the channel can corrupt the signal, and thus it is possible that the received signal $s_{\mathcal{Y}}$ is not the same as the transmitted signal $s_{\mathcal{X}}$. In particular, the *transition matrix* below gives the (conditional) probability that the receiver hears $s_{\mathcal{Y}}$ when the transmitter sends $s_{\mathcal{X}}$.

Transmitted	Received $\mathcal{Y}$		
$\mathcal{X}$	1	2	3
1	0.8	0.1	0.1
2	0.05	0.9	0.05
3	0.15	0.05	0.8

For example, this table is saying that a transmitted s_1 is received as an s_1, or s_2 or s_3 with probabilities $0.8, 0.1$, and 0.1 respectively.

(a) Suppose that $\mathcal{X}$ has pmf $p_{\mathcal{X}}(1) = 0.5$, $p_{\mathcal{X}}(2) = 0.25$, $p_{\mathcal{X}}(3) = 0.25$. What is the pmf of $\mathcal{Y}$?

(b) Given that the receiver heard $\mathcal{Y} = 3$, what are the *conditional* probabilities of $\{\mathcal{X} = 1\}$? $\{\mathcal{X} = 2\}$? $\{\mathcal{X} = 3\}$?

7. An urn contains r red and g green balls. Two balls are drawn at random from the urn, with the first ball being returned to the urn (which is then well shaken) before the second ball is drawn. Let R_1 and R_2 respectively denote the events that the first and second balls are red.

(a) What are $P(R_1)$ and $P(R_2)$?

(b) Now suppose that when the first ball is returned to the urn, c *additional* balls of the *same color* are also put into the urn (which is then well shaken before the second ball is drawn.) Clearly $P(R_1)$ is the same as before, but what is $P(R_2)$ now? Remember that the urn now contains $r+g+c$ balls. Simplify your answer and compare to the value of $P(R_2)$ that you obtained in part (a).

(c) For the experiment of part (b), what is the conditional probability that the urn contained $r+c$ red balls given that R_2 occurred?

8. We consider Problem 6 of the previous problem set under the condition that 15 of the 105 passengers are arriving in Chicago on a connecting flight. If the connecting flight is on time, *all* show up at the gate for the flight to Champaign: else none of the 15 make it. The remaining 90 passengers make up their minds individually as before (and independently of the fate of the incoming flight). The number of *these* who show up at the gate is thus a binomial random variable $\mathcal{Z}$ with parameters $(90, 0.9)$. As before, let $\mathcal{X}$ denote the number of passengers presenting themselves at the gate. Let T denote the event that the connecting flight is on time, and $A = \{\mathcal{X} \leq 100\}$ the event that all who show up at the gate get to board the flight.

(a) What are the conditional probabilities $P\{A|T\}$ and $P\{A|T^c\}$?

(b) Suppose that $P(T) = \frac{2}{3}$. Find $P(A)$.

(c) Given that all who showed up got a seat, what is the conditional probability that the connecting flight was on time?