University of Illinois Fall 2006

ECE 413: Problem Set 4

Due:	Wednesday September 20 at the beginning of class.
Reading:	Ross, Chapter 4
Noncredit Exercises:	DO NOT turn these in.
	Chapter 4: Problems 33, 38, 39, 40-43, 48, 51-59;
	Theoretical Exercises 16-19; Self-Test Problems 9, 13, 15, 16.

This Problem Set contains six problems

1. Use a spreadsheet/Mathematica/MATLAB for this problem.
   Let A denote an event of probability p.

   (a) For $p = 0.1, 0.25, 0.4, 0.5, 0.6, 0.75,$ and 0.9, find the numerical values of the probabilities that A occurs $0, 1, 2, \ldots, 10$ times on 10 trials.

   (b) You have computed the pmf of a binomial random variable $\mathcal{X}_p$ with parameters $(10, p)$ for seven choices of p. For each value of p, draw a bar graph of the pmf of $\mathcal{X}_p$. (The pmf of $\mathcal{X}_{0.5}$ is shown on page 157 of the text!).

   (c) What is the relationship between the pmfs of $\mathcal{X}_p$ and $\mathcal{X}_{1-p}$?

2. Let $\mathcal{X}$ denote a binomial random variable with parameters (N, p).

   (a) Show that $\mathcal{Y} = N - \mathcal{X}$ is a binomial random variable with parameters $(N, 1-p)$.

   (b) What is $P\{\mathcal{X} \text{ is even}\}$? Hint: Use the binomial theorem to write an expression for $(x+y)^n + (x-y)^n$ and then set $x = 1-p$, $y = p$.

3. Eight persons have purchased tickets (\$$F$ per person) for travel in a 5-passenger plane on a scheduled airline flight in Ruritania. The number of persons who actually show up to travel can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(8, 0.5)$. Naturally, if more than 5 persons show up, only 5 get to go and the rest are left behind. Let $\mathcal{Y}$ denote the number of persons left behind.

   (a) What is $\mathsf{E}[\mathcal{X}]$?

   (b) Find the pmf of $\mathcal{Y}$.

   (c) What is $\mathsf{E}[\mathcal{Y}]$? Calculate this in two ways: (i) from your answer to part (b), and (ii) by using the fact that $\mathcal{Y}$ is a function of $\mathcal{X}$, and hence LOTUS allows us to calculate $\mathsf{E}[\mathcal{Y}]$ directly from the pmf of $\mathcal{X}$.

   (d) The flight costs the airline \$200 plus \$10 for each passenger carried on board. (Even if no passengers show up, the flight must still go because the plane is needed at the destination for use in the return flight). According to Ruritanian Aviation Administration (RAA) rules, passengers who don't show up are SOL; they cannot use their tickets on another flight and they cannot get a refund either. On the other hand, each bumped passenger gets a full refund of \$$F$ plus \$20 as compensation for being denied boarding. Let $\mathcal{Z}$ denote the net profit to the airline from this flight. Use LOTUS to express $\mathsf{E}[\mathcal{Z}]$ as a function of F and determine the value of F for which the profit is exactly 0, i.e. the break-even point.

   (e) What is the pmf of $\mathcal{Z}$?

4. In the game of Chuck-A-Luck played at fairs and carnivals in the MidWest, bets are placed on numbers 1, 2, 3, 4, 5, 6, and then three fair dice are rolled. If the number chosen does not show up on any of the three dice, the bettor loses his stake. Otherwise, the dealer pays the bettor one or two or three times the amount staked according as the number chosen shows up on one or two or all three of the dice. Of course, the amount of the bet is also returned to the bettor but is *not* counted as part of the *winnings* from this game. Let $\mathcal{X}$ denote the winnings in this game for a \$6 bet, and remember that negative values of $\mathcal{X}$ correspond to losses.

   (a) What are the values taken on by $\mathcal{X}$?

   (b) What is the pmf of $\mathcal{X}$?

   (c) What is the value of $\mathsf{E}[X]$?

   (d) A player splits his \$6 bet and wagers \$1 on each of the six numbers. Let $\mathcal{Y}$ denote the winnings of this player. Repeat parts (a)-(c) for $\mathcal{Y}$. Does the splitting strategy improve the average winnings in this game?

5. Let $\mathcal{Y}$ denote a Poisson random variable with parameter λ.

   (a) Show that $P\{\mathcal{Y} \text{ is even}\} = \exp(-\lambda)\cosh(\lambda)$.

   (b) In Problem 2 of this Problem Set, you proved (I hope!) that the probability that a binomial random variable with parameters (N, p) is *even* is $[1+(1-2p)^N]/2$. Now, for large N and small p, the binomial probability $P\{\mathcal{X} = k\}$ is well approximated by the Poisson probability $\exp(-Np)(Np)^k/k!$. Under the same conditions, show that $[1 + (1 - 2p)^N]/2 \approx \exp(-Np)\cosh(Np)$ and thus your answer of part (a) is consistent with the previous result.

   (c) Now suppose that the value of λ is unknown. The experiment is performed and it is observed that $\mathcal{Y} = k$. What is the *likelihood* of this observation? What is the *maximum likelihood* estimate $\hat{\lambda}$ of λ? That is, what choice of positive number $\hat{\lambda}$ maximizes the likelihood of the observation $\mathcal{Y} = k$?

6. Suppose that 105 passengers hold reservations for a 100-passenger flight from Chicago to Champaign. The number of passengers who show up at the gate can be modeled as a binomial random variable $\mathcal{X}$ with parameters $(105, 0.9)$.

   (a) On average, how many passengers show up at the gate?

   (b) If $\mathcal{X} \leq 100$, everyone who shows up gets to go. Find the value of $P\{\mathcal{X} \leq 100\}$.

   (c) Explain why the number of *no-shows* can be modeled as a binomial random variable $\mathcal{Y}$ with parameters $(105, 0.1)$.

   (d) Notice that the probability that everyone who shows up gets to go can also be expressed as $P\{\mathcal{Y} \geq 5\}$. Use the *Poisson approximation* to compute $P\{\mathcal{Y} \geq 5\}$ and compare your answer to the "more exact" answer that you found in part (b).