University of Illinois Fall 2006

ECE 413: Final Examination: Problems and Solutions: 3 hours

1. A, B, and C are independent events of probabilities 0.3, 0.4 and 0.5 respectively. Find $P(A\cup(B\cap C))$ and $P(A\cup B|B\oplus C)$.

   $P(A\cup(B\cap C)) = P(A)+P(B\cap C)-P(A\cap B\cap C) = P(A)+P(B)P(C)-P(A)P(B)P(C)$ since the events are independent. Hence, $P(A\cup(B\cap C)) = 0.3+0.4\cdot 0.5-0.3\cdot 0.4\cdot 0.5 = 0.44$.

   $$P(A\cup B|B\oplus C) = \frac{P((A\cup B)\cap(BC^c\cup B^cC))}{P(BC^c\cup B^cC)} = \frac{P(ABC^c\cup BC^c\cup AB^cC)}{P(B)P(C^c)+P(B^c)P(C)} = \frac{P(BC^c\cup AB^cC)}{P(B)P(C^c)+P(B^c)P(C)}$$
   $$= \frac{0.4\times 0.5+0.3\times 0.6\times 0.5}{0.4\times 0.5+0.6\times 0.5} = \frac{0.29}{0.5} = 0.58.$$

2. $\mathcal{X}$ denotes a Poisson random variable with parameter $\ln(3)$. Find the numerical values of the mean and variance of $\mathcal{Y} = \cos(\pi\mathcal{X})$.

   Since $\mathcal{X}$ takes on integer values, $\mathcal{Y}$ has value $+1$ or -1 according as $\mathcal{X}$ is even or odd, that is, $\mathcal{Y} = (-1)^{\mathcal{X}}$. Hence,
   $$\mathsf{E}[\mathcal{Y}] = \sum_{k=0}^{\infty}(-1)^k\exp(-\lambda)\frac{\lambda^k}{k!} = \exp(-\lambda)\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = \exp(-2\lambda) = \frac{1}{9} \text{ since } \lambda = \ln(3).$$
   Next, note that $\mathsf{E}[\mathcal{Y}^2] = \sum_{k=0}^{\infty}\exp(-\lambda)\frac{\lambda^k}{k!} = 1$ and hence $\mathsf{var}(\mathcal{Y}) = \mathsf{E}[\mathcal{Y}^2]-(\mathsf{E}[\mathcal{Y}])^2 = 1-\frac{1}{81} = \frac{80}{81}$.

3. $\mathcal{X}$ denotes a continuous random variable with pdf $f_{\mathcal{X}}(u)$ satisfying $f_{\mathcal{X}}(u) = f_{\mathcal{X}}(-u)$ for all u, $-\infty < u < \infty$. Suppose that $\mathsf{var}(\mathcal{X}) = 9$. Let $\mathcal{Y} = |\mathcal{X}|$ and $\mathcal{Z} = -\mathcal{X}$, and consider the statements below for all random variables satisfying these conditions.
   Mark ALWAYS if the statement is *true for all* such random variables; mark NEVER if the statement is *false for all* such random variables; and mark MAYBE if the statement is *true for some* such random variables *but not all* such random variables.

ALWAYS	NEVER	MAYBE	
■	□	□	$P\{\mathcal{X}>\alpha\} = F_{\mathcal{X}}(-\alpha)$ for all α, $-\infty<\alpha<\infty$. Always true by symmetry of pdf: $P\{\mathcal{X}>\alpha\} = P\{\mathcal{X}<-\alpha\} = F_{\mathcal{X}}(-\alpha)$
■	□	□	$F_{\mathcal{Y}}(v) = 2F_{\mathcal{X}}(v)-1$ for $v\ge 0$, and 0 for $v<0$. Always true by symmetry of pdf: $P\{\lvert\mathcal{X}\rvert<v\} = P\{-v<\mathcal{X}<v\} = F_{\mathcal{X}}(v)-F_{\mathcal{X}}(-v)$
□	■	□	$F_{\mathcal{Z}}(w) = F_{\mathcal{X}}(-w)$ for all w, $-\infty<w<\infty$. Always false: $F_{\mathcal{X}}(-w)$ is a decreasing function of w and cannot be a valid CDF.
■	□	□	$f_{\mathcal{Z}}(w) = f_{\mathcal{X}}(-w)$ for all w, $-\infty<w<\infty$. Always true: $F_{\mathcal{Z}}(w) = 1-F_{\mathcal{X}}(-w)$ has derivative $f_{\mathcal{X}}(-w)$.
■	□	□	$\mathsf{E}[\mathcal{Y}^2] = 9$. Always true: $\mathsf{E}[\mathcal{Y}^2] = \mathsf{E}[\mathcal{X}^2] = \mathsf{var}(\mathcal{X})+(\mathsf{E}[\mathcal{X}])^2 = 9$ since $\mathsf{E}[\mathcal{X}] = 0$.
□	■	□	$\mathsf{E}[\mathcal{Y}] = 3$. Always false: it would imply that $\mathsf{var}(\mathcal{Y}) = 0$ which is not true.
■	□	□	$\mathsf{var}(\mathcal{Y}) < 9$. Always true: $\mathsf{var}(\mathcal{Y}) = \mathsf{E}[\mathcal{Y}^2]-(\mathsf{E}[\mathcal{Y}])^2 = 9-(\mathsf{E}[\mathcal{Y}])^2 < 9$. since $0<\mathsf{E}[\mathcal{Y}]<3$.
□	□	■	$\mathsf{var}(\mathcal{Y}) < 8$. Maybe true: e.g. when $\mathcal{X}\sim\mathsf{Uniform}(-3\sqrt{3},3\sqrt{3})$, $\mathsf{var}(\mathcal{Y}) = 2.25$.
■	□	□	$\mathsf{E}[\mathcal{X}\mathcal{Y}] = 0$. Always true! By LOTUS, $\mathsf{E}[\mathcal{X}\mathcal{Y}] = \int_0^{\infty}u^2f_{\mathcal{X}}(u)\,du+\int_{-\infty}^{0}u(-u)f_{\mathcal{X}}(u)\,du = 0$.
■	□	□	$\mathcal{X}$ and $\mathcal{Y}$ are uncorrelated random variables. Always true: $\mathsf{cov}(\mathcal{X},\mathcal{Y}) = \mathsf{E}[\mathcal{X}\mathcal{Y}]-\mathsf{E}[\mathcal{X}]\mathsf{E}[\mathcal{Y}] = 0$ since $\mathsf{E}[\mathcal{X}\mathcal{Y}] = \mathsf{E}[\mathcal{X}] = 0$.
□	■	□	$\mathcal{X}$ and $\mathcal{Y}$ are independent random variables. Always false: knowing the value of $\mathcal{X}$ tells us the exact value of $\mathcal{Y}$.
■	□	□	$F_{\mathcal{X}}(6) \ge \frac{7}{8} = 0.8750$. Always true: from Chebyshev's inequality, $P\{\lvert\mathcal{X}\rvert>6\}\le 0.25 \Rightarrow F_{\mathcal{X}}(6)\ge 1-\frac{0.25}{2}$.
□	□	■	$F_{\mathcal{X}}(6) \ge 0.9772$. Maybe true: e.g. if $\mathcal{X}\sim\mathcal{N}(0,9)$, then $F_{\mathcal{X}}(6) = 0.9772$.

4. A radio-frequency signal is either a radar echo (hypothesis H_1) or ambient noise (hypothesis H_0). The *phase* of the signal is modeled as a continuous random variable $\mathcal{X}$ whose pdf is as follows:

- When H_0 is true, $\mathcal{X}$ has pdf $f_0(u) = \begin{cases} \frac{1}{2\pi}, & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$
- When H_1 is true, $\mathcal{X}$ has pdf $f_1(u) = \begin{cases} \frac{1}{2\pi}\left(1 + \cos u\right), & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$

The radar receiver measures $\mathcal{X}$ and decides which hypothesis is true.

(a) Suppose that the *maximum-likelihood* decision rule is being used. What value(s) of $\mathcal{X}$ result in a decision in favor of H_1?

The likelihood ratio is $\Lambda(u) = \frac{f_1(u)}{f_0(u)} = 1 + \cos u > 1$ if and only if $\mathcal{X} \in (-\pi/2, \pi/2) = \Gamma_1$.

(b) Find the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

$$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi}\,du = \frac{1}{2}.$$

$$P_{\text{MD}} = 1 - \int_{\Gamma_1} f_1(u)\,du = 1 - \int_{-\pi/2}^{\pi/2} \frac{1}{2\pi}(1 + \cos u)\,du = \frac{1}{2} - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos u\,du = \frac{1}{2} - \frac{1}{2\pi}\sin u\Big|_{-\pi/2}^{\pi/2} = \frac{\pi - 2}{2\pi}.$$

(c) Now suppose that $P(\mathsf{H}_0) = \pi_0 = \frac{1}{3}$, $P(\mathsf{H}_1) = \pi_1 = \frac{2}{3}$. What is the *average* error probability $\bar{P}_e$ of the maximum *a posteriori* probability (MAP) (that is, minimum-error-probability or Bayesian) decision rule?

In this case, $\Gamma_1 = \{u : \Lambda(u) > \pi_0/\pi_1\} = \{u : 1 + \cos u > 1/2\} = \{u : \cos u > -1/2\} = (-2\pi/3, 2\pi/3)$ and hence

$$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-2\pi/3}^{2\pi/3} \frac{1}{2\pi}\,du = \frac{2}{3} \text{ while}$$

$$P_{\text{MD}} = 1 - \int_{-2\pi/3}^{2\pi/3} \frac{1}{2\pi}(1 + \cos u)\,du = \frac{1}{3} - \frac{1}{2\pi}\int_{-2\pi/3}^{2\pi/3} \cos u\,du = \frac{1}{3} - \frac{1}{2\pi}\sin u\Big|_{-2\pi/3}^{2\pi/3} = \frac{1}{3} - \frac{\sqrt{3}}{2\pi}.$$

Therefore, $\bar{P}_e = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{2}{3}\left[\frac{1}{3} - \frac{\sqrt{3}}{2\pi}\right] = \frac{4}{9} - \frac{1}{\pi\sqrt{3}}.$

(d) For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_0 regardless of the value of $\mathcal{X}$?

$\Lambda(u) = 1 + \cos u$ has maximum value 2 and is thus always footnotesize er than the threshold $\frac{\pi_0}{\pi_1}$ if $\pi_0 > \frac{2}{3}$.

5. A professor breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.1$ per minute.

(a) What is the expected length of time between two successive chalk breaks?

The interarrival time is exponentially distributed with parameter $\lambda = 0.1$ and has mean $\lambda^{-1} = 10$ minutes.

(b) What is the average number of times that the professor breaks the chalk during a 50 minute lecture?

The number of arrivals in 50 minutes is $\mathsf{Poisson}(50\lambda)$ and thus has mean $50\lambda = 5$.

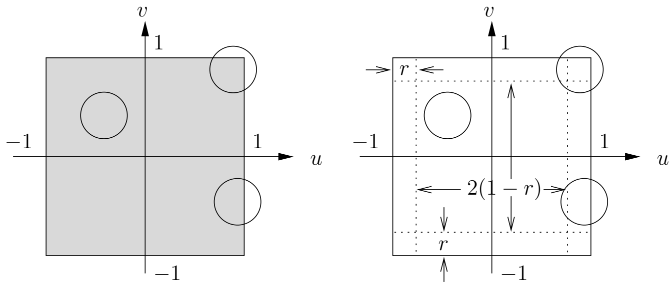
(c) *Given* that the professor broke 6 chalk pieces in 50 minutes, what is the average number of pieces he broke in the first 25 minutes?

Conditioned on $N_{(0,50]} = 6$, $N_{(0,25]} \sim \mathsf{Binomial}(6, \frac{25}{50}) = \mathsf{Binomial}(6, \frac{1}{2})$ whose mean is 3.
This binomial distribution can also be derived directly as follow. For $0 \le k \le 6$,

$$P(\{N_{(0,25]} = k\}|\{N_{(0,50]} = 6\}) = \frac{P(\{N_{(0,25]} = k\} \cap \{N_{(0,50]} = 6\})}{P(\{N_{(0,50]} = 6\})} = \frac{P(\{N_{(0,25]} = k\} \cap \{N_{(25,50]} = 6 - k\})}{P(\{N_{(0,50]} = 6\})}$$

$$= \frac{\exp(-2.5)\frac{(2.5)^k}{k!}\exp(-2.5)\frac{(2.5)^{6-k}}{(6-k)!}}{\exp(-5)\frac{5^6}{6!}} = \frac{6!}{k!(6-k)!}\left(\frac{2.5}{5}\right)^k\left(\frac{2.5}{5}\right)^{6-k} = \binom{6}{k}\left(\frac{1}{2}\right)^6 = \mathsf{Binomial}(6, \tfrac{1}{2}; k).$$

6. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the interior of a square of side 2 centered at the origin. Consider a circle of radius $r < 1$ centered at $(\mathcal{X}, \mathcal{Y})$, and let $\mathcal{Z} \in \{0, 1, 2\}$ denote the number of sides of the square that are crossed by the circle, as illustrated in the figure below.



If $\mathsf{E}[\mathcal{Z}] = \dfrac{3}{2}$, what is the value of r?

The joint pdf has value 4 on the shaded region. Now, $\mathcal{Z} = 2$ or 1 or 0 according as $(\mathcal{X}, \mathcal{Y})$ is respectively in one of the four $r \times r$ corner squares, or one of the four $r \times 2(1-r)$ edge rectangles, or the $2(1-r) \times 2(1-r)$ central square shown above in the sketch on the right. Hence, $P(\mathcal{Z} = 2) = r^2, P(\mathcal{Z} = 1) = 2r(1-r), P(\mathcal{Z} = 0) = (1-r)^2$, that is, $\mathcal{Z} \sim \mathsf{Binomial}(2, r)$. Since $\mathsf{E}[\mathcal{Z}] = 2r = \frac{3}{2}$, we get $r = \frac{3}{4}$.

7. The joint pdf of $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u, v) = \begin{cases} 2u, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Are $\mathcal{X}$ and $\mathcal{Y}$ independent random variables? SHOW YOUR WORK.

Yes, $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables since their marginal pdfs are easily found to be $f_{\mathcal{X}}(u) = 2u$ for $0 < u < 1$ and $f_{\mathcal{X}}(v) = 1$ for $0 < v < 1$, and $f_{\mathcal{X},\mathcal{Y}}(u, v) = f_{\mathcal{X}}(u) f_{\mathcal{X}}(v)$ for all u, v.

(b) Find the pdf of $\mathcal{Z} = \mathcal{X}\mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.

$\mathcal{Z}$ takes on values in $(0, 1)$. For any $\alpha,\ 0 < \alpha < 1$,

$$P\{\mathcal{Z} > \alpha\} = \int_{v=\alpha}^{1} \int_{u=\alpha/v}^{1} 2u\, du\, dv = \int_{\alpha}^{1} 1 - \frac{\alpha^2}{v^2}\, dv = v + \left.\frac{\alpha^2}{v}\right|_{\alpha}^{1} = 1 + \alpha^2 - \alpha - \alpha = (1-\alpha)^2.$$

Hence, $f_{\mathcal{Z}}(\alpha) = -\dfrac{d}{d\alpha} P\{\mathcal{Z} > \alpha\} = \begin{cases} 2(1-\alpha), & 0 < \alpha < 1, \\ 0, & \text{otherwise.} \end{cases}$ It is easily verified that this is a valid pdf.

8. The jointly Gaussian random variables $\mathcal{X}$ and $\mathcal{Y}$ have means 0 and 14 respectively, variances 4 and 16 respectively, and correlation coefficient $\frac{1}{16}$.

(a) Find the pdf of the random variable $\mathcal{Z} = 5\mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all $\alpha, -\infty < \alpha < \infty$.

$\mathcal{Z}$ is a linear function of jointly Gaussian random variables and hence is itself a Gaussian random variable with mean $\mathsf{E}[\mathcal{Z}] = \mathsf{E}[5\mathcal{X} + \mathcal{Y}] = 14$ and variance $\sigma_{\mathcal{Z}}^2 = 5^2 \cdot \sigma_{\mathcal{X}}^2 + 1^2 \cdot \sigma_{\mathcal{Y}}^2 + 2 \cdot 5 \cdot 1 \cdot \rho\sigma_{\mathcal{X}}\sigma_{\mathcal{Y}} = 100 + 16 + 5 = 121$. Hence, $f_{\mathcal{Z}}(\alpha) = (11\sqrt{2\pi})^{-1/2} \exp(-(\alpha - 14)^2/242), \ -\infty < \alpha < \infty$.

(b) Find the numerical value of $P\{\mathcal{Y} > 3\mathcal{X}\}$.

$P\{\mathcal{Y} > 3\mathcal{X}\} = P\{3\mathcal{X} - \mathcal{Y} < 0\}$ where $3\mathcal{X} - \mathcal{Y}$ is a Gaussian random variable with mean $3 \cdot 0 - 14 = -14$ and variance $3^2 \cdot \sigma_{\mathcal{X}}^2 + 1^2 \cdot \sigma_{\mathcal{Y}}^2 - 2 \cdot 3 \cdot 1 \cdot \rho\sigma_{\mathcal{X}}\sigma_{\mathcal{Y}} = 36 + 16 - 3 = 49$. Hence, $P\{\mathcal{Y} > 3\mathcal{X}\} = \Phi(0 - (-14))/7) = \Phi(2) = 0.9772$.