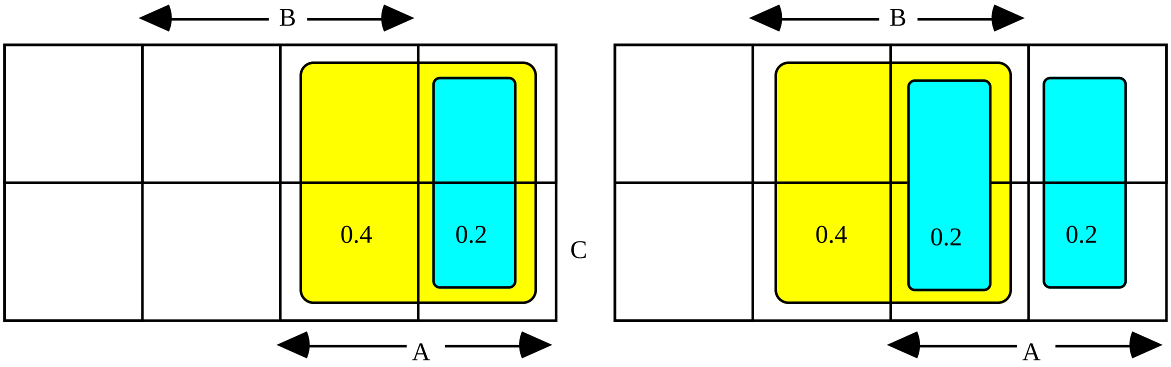
ECE 413: Solutions to Hour Exam I

1. [20 points] Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = P(B) = 0.4$, $P(C) = 0.5$, and $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$. Find the following probabilities: $P(A \cap B)$, $P(A^c \cap B)$, $P((A \cup B \cup C)^c)$, and $P(C^c | (A^c \cap B^c))$.

1. The Karnaugh maps shown below are very useful in visualizing the problem.



We get that

- $P(A \cap B) = P(A) - P(A \cap B^c) = 0.4 - 0.2 = 0.4$. Note that $P(A \cup B) = 0.6$.
- $P(A^c \cap B) = P(B) - P(A \cap B) = 0.4 - 0.2 = 0.2$.
- $P((A \cup B \cup C)^c) = 1 - P(A \cup B \cup C) = 1 - [P(A \cup B) + P(A^c \cap B^c \cap C)] = 1 - [0.6 + 0.2] = 0.2$. Note that $P((A \cup B \cup C)^c) = P(A^c \cap B^c \cap C^c)$, and also that $P(A^c \cap B^c) = 1 - P(A \cup B) = 0.4$.
- $P(C^c | (A^c \cap B^c)) = \dfrac{P(A^c \cap B^c \cap C^c)}{P(A^c \cap B^c)} = \dfrac{0.2}{0.4} = 0.5$.

2. [24 points] A bag contains $n \geq 2$ pairs of shoes in distinct styles and sizes.
   (a) You pick two shoes at random from the bag. Note that this is sampling without replacement.
      i. What is the probability that you get a pair of shoes?
      ii. What is the probability of getting one left shoe and one right shoe?
   (b) You now pick a third shoe at random from the bag *without* returning the two shoes that you previously picked to the bag.
      i. What is the probability that you have a pair of shoes among the three that you have picked?
      ii. What is the probability that there is at least one left shoe and at least one right shoe among the three?

2. (a) i. There are $\binom{2n}{2}$ choices of two shoes from $2n$ of which only n choices yield a pair. Hence,

$$P(\text{pair}) = \frac{n}{\binom{2n}{2}} = \frac{n}{2n(2n-1)/1 \times 2} = \frac{1}{2n-1}.$$

More simply, think of choosing the shoes sequentially. Regardless of what the first shoe drawn is, the chance of getting its mate on the second draw is $\dfrac{1}{2n-1}$. Note that this has value 1 if $n = 1$, which makes perfect sense.

ii. Any one of the n left shoes can be paired with any one of the n right shoes. So, n^2 of the $\binom{2n}{2}$ choices yield one left shoe and one right shoe in the two drawn, giving that

$$P(\text{one L, one R }) = \frac{n^2}{\binom{2n}{2}} = \frac{n^2}{2n(2n-1)/1 \times 2} = \frac{n}{2n-1}.$$

Again, more simply, regardless of what the first shoe is, the chances of getting a shoe of the opposite footality on the second draw is $\frac{n}{2n-1}$. Note that this has value 1 if $n = 1$, which makes perfect sense.

(b) i. Any of the n pairs and any of the other $2n - 2$ shoes form a set of 3 shoes. Hence, $P(\text{pair among three}) = \frac{n(2n-2)}{\binom{2n}{3}} = \frac{n(2n-2)}{2n(2n-1)(2n-2)/1 \times 2 \times 3} = \frac{3}{2n-1}$.
Alternatively, conditioned on the first two shoes being a pair, the probability of the three shoes including a pair is 1 (Duh!). If the first two shoes *do not* constitute a pair, then the chances of getting a match on the next draw are $\frac{2}{2n-2}$. Hence,

$$P(\text{pair among three}) = \frac{1}{2n-1} \times 1 + \frac{2n-2}{2n-1} \times \frac{2}{2n-2} = \frac{3}{2n-1}.$$

Note that this has value 1 when $n = 2$, which makes perfect sense.

ii. The probability that all three shoes are left shoes is $\frac{\binom{n}{3}}{\binom{2n}{3}} = \frac{n(n-1)(n-2)}{2n(2n-1)(2n-2)} = \frac{n-2}{8n-4}$.
Similarly for all three shoes being right shoes. Hence,
$P(\text{one L and one R among three}) = 1 - 2 \times \frac{n-2}{8n-4} = \frac{3n}{4n-2}$.
Alternatively, conditioned on the first two shoes being a left shoe and a right shoe, the probability of having a left shoe and a right shoe among the three is 1, while if the first two shoes are *not* a left and a right shoe, the chances of getting the missing footality on the next draw is $\frac{n}{2n-2}$. Hence,

$$P(\text{one L and one R among three}) = \frac{n}{2n-1} \times 1 + \frac{n-1}{2n-1} \times \frac{n}{2n-2} = \frac{3n}{4n-2}.$$

Note that this has value 1 when $n = 2$, which makes perfect sense.

3. [36 points] Fred and Wilma take turns tossing a coin with $P(\text{Heads}) = p$ and $P(\text{Tails}) = q = 1 - p$. Fred tosses first, then Wilma, then Fred again, and so on until the game ends. Let F_i and W_i respectively denote the events that Fred and Wilma win the i-th game.

(a) The rules are that the first one to toss a Head wins the game. In succeeding games, the *loser* of the previous game tosses first. Remember that Fred tosses first in the first game.

i. Find $P(F_1)$. Which is larger: $P(F_1)$ or $P(W_1)$?

ii. What is the probability that Fred wins the second game? Which is larger, $P(F_2)$ or $P(W_2)$?

(b) In a different game, the rule is that the first one to *match* the previous toss wins the game. Wilma graciously insists that Fred go first as usual in the first game, and the schmuck accepts without realizing that he has no previous toss to match! Note that there are at least two coin tosses in the first game, and that HH or TT constitute a win for Wilma, while HTT and THH constitute a win for Fred, etc.

i. Find $P(F_1)$. Which is larger: $P(F_1)$ or $P(W_1)$?

ii. The *first toss* of the second game is the *winning* toss of the first game. Thus, outcomes HHH or TTT constitute a win for Wilma (her first toss matches Fred's first toss) in the first game *and* a win for Fred in the second game (his second toss matches Wilma's previous winning toss). What is the probability that Fred wins the second game? Which is larger, $P(F_2)$ or $P(W_2)$?

3. (a) i. Fred wins the first game if the outcome is H, or TTH, or TTTTH, or .... Hence,

$$P(F_1) = p + q^2p + q^4p + \cdots = \frac{p}{1-q^2} = \frac{p}{(1-q)(1+q)} = \frac{1}{1+q}.$$

Wilma wins the first game if the outcome is TH, or TTTH, or TTTTTH, or .... Hence,

$$P(W_1) = qp + q^3p + q^5p + \cdots = \frac{pq}{1-q^2} = \frac{pq}{(1-q)(1+q)} = \frac{q}{1+q} = qP(F_1) < P(F_1).$$

Alternatively, condition on the result of the first toss. Clearly, $P(F_1|H) = 1, P(W_1|H) = 0$. On the other hand, if the first toss is a T, then effectively Wilma is going first! That is, she can win outright by tossing a H, etc. Hence $P(W_1|T) = P(F_1)$ while $P(F_1|T) = P(W_1)$, giving

$$\begin{aligned} P(F_1) &= P(F_1|H)P(H) + P(F_1|T)P(T) &= 1 \times p + P(W_1) \times q &= p + qP(W_1) \\ P(W_1) &= P(W_1|H)P(H) + P(W_1|T)P(T) &= 0 \times p + P(F_1) \times q &= qP(F_1) \end{aligned}$$

giving $P(F_1) = \frac{p}{1-q^2} = \frac{1}{1+q}$ and $P(W_1) = qP(F_1) = \frac{q}{1+q}$ as before.

ii. The person tossing first in a game has probability $1/(1+q)$ of winning the game. Since the loser of the first game tosses first in the second game, we have that

$$P(F_2|F_1) = \frac{q}{1+q}, \quad P(F_2|W_1) = \frac{1}{1+q}, \quad P(W_2|F_1) = \frac{1}{1+q}, \quad P(W_2|W_1) = \frac{q}{1+q}$$

giving

$$\begin{aligned} P(F_2) &= P(F_1|F_1)P(F_1) + P(F_1|W_1)P(W_1) &= \tfrac{q}{1+q} \times \tfrac{1}{1+q} + \tfrac{1}{1+q} \times \tfrac{q}{1+q} &= \tfrac{2q}{(1+q)^2} \\ P(W_2) &= P(W_2|F_1)P(F_1) + P(W_2|W_1)P(W_1) &= \tfrac{1}{1+q} \times \tfrac{1}{1+q} + \tfrac{q}{1+q} \times \tfrac{q}{1+q} &= \tfrac{1+q^2}{(1+q)^2} \end{aligned}$$

Since $1+q^2 > 2q$ (why?), $P(W_2) > P(F_2)$. Recall that Fred had a better chance of winning the first game, but now the situation has reversed. The chances are also more equally balanced.

(b) i. Fred wins the first game if the outcome is HTT, or HTHTT or HTHTHTT, or ...or THH, or THTHH, or THTHTHH, or .... Hence,

$$P(F_1) = (pq)q + (pq)^2q + (pq)^3q + \cdots \quad + \quad (pq)p + (pq)^2p + (pq)^3p + \ldots = \frac{pq}{1-pq}.$$

Wilma wins the first game if the outcome is HH, or HTHH, or HTHTHH, or ...or TT, or THTT, or THTHTT, or ldots. Hence,

$$P(W_1) = p^2 + (pq)p^2 + (pq)^2p^2 + \cdots \quad + \quad q^2 + (pq)q^2 + (pq)^2q^2 + \ldots = \frac{p^2+q^2}{1-pq}.$$

Note that $P(F_1)+P(W_1) = 1$ since $p^2+q^2+pq = 1-p+p^2 = 1-pq$. Also, $P(W_1) > P(F_1)$.

ii. The person tossing first in a game has probability $pq/(1-pq)$ of winning the game. Since the winner of the first game effectively gets to toss first in the second game, we have that

$$P(F_2|F_1) = \frac{pq}{1-pq}, \quad P(F_2|W_1) = \frac{p^2+q^2}{1-pq}, \quad P(W_2|F_1) = \frac{p^2+q^2}{1-pq}, \quad P(W_2|W_1) = \frac{pq}{1-pq}$$

giving

$$\begin{aligned} P(F_2) &= \frac{pq}{1-pq} \times \frac{pq}{1-pq} + \frac{p^2+q^2}{1-pq} \times \frac{p^2+q^2}{1-pq} &= \frac{(p^2+q^2)^2 + (pq)^2}{(1-pq)^2} \\ P(W_2) &= \frac{p^2+q^2}{1-pq} \times \frac{pq}{1-pq} + \frac{pq}{1-pq} \times \frac{p^2+q^2}{1-pq} &= \frac{2pq(p^2+q^2)}{(1-pq)^2} \end{aligned}$$

Since $a^2+b^2 > 2ab$, setting $a = p^2+q^2$ and $b = pq$, we see that $P(F_2) > P(W_2)$. Recall that Wilma had a better chance of winning the first game, but now the situation has reversed. As before, the chances are also more equally balanced in the second game.

4. (a) [6 points] Find $\mathsf{E}[\mathcal{X}^2]$ for a Poisson random variable $\mathcal{X}$ with mean 5.

   (b) [6 points] If $\mathcal{Y}$ is a geometric random variable with mean 4, what is $\mathsf{var}(2 - 3\mathcal{Y})$?

   (c) [8 points] If $\mathcal{Z}$ denotes the number of occurrences of an event of probability p on 10 independent trials, what is the *conditional* expected value of $\mathcal{Z}$ given that the event occurred 4 times on the first six trials?

4. (a) Since a Poisson random variable has mean and variance both equal to its parameter λ, and we are given that $\lambda = 5$ in this case, we get that $\mathsf{E}[\mathcal{X}^2] = \mathsf{var}(\mathcal{X}) + (\mathsf{E}[\mathcal{X}])^2 = 5^2 + 5 = 30$.

   (b) Since a geometric random variable has mean $1/p$ and variance $(1-p)/p^2$, and we are given that $1/p = 2$ in this case, we get that $\mathsf{var}(2 - 3\mathcal{Y}) = 3^2\mathsf{var}(\mathcal{Y}) = 9 \cdot \dfrac{1/2}{1/4} = 18$.

   (c) What happens on the last 4 trials is independent of what happens on the first 6. Thus, the number of occurrences of the event on the last 4 trials is a binomial random variable with parameters $(4, p)$, and hence expected value $4p$. This is true regardless of what happened on the first six trials. Therefore the conditional expectation of $\mathcal{Z}$ is $4 + 4p$.