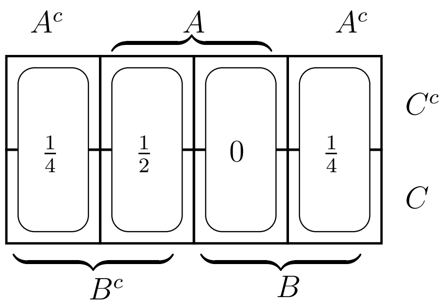
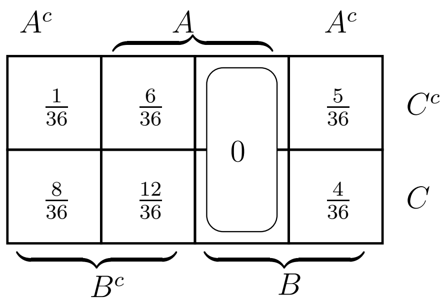
ECE 413: Solutions to Hour Exam II

1. Since A and B are mutually exclusive events, we readily get the Karnaugh map shown on the left. Since A and C are independent events, we have that $P(A \cap C) = P(A)P(C) = \frac{1}{3}$. Note also that $P(A \cap C) = P(A \cap B \cap C) + P(A \cap B^c \cap C) = 0 + P(A \cap B^c \cap C)$, and so $P(A \cap B^c \cap C) = \frac{1}{3}$. Next, $P(B \cap C) = P(B|C)P(C) = \frac{1}{9} = P(A^c \cap B \cap C)$. Finally, from $P(C) = \frac{2}{3} = P(A \cap B^c \cap C) + P(A^c \cap B \cap C) + P(A^c \cap B^c \cap C)$, we get that $P(A^c \cap B^c \cap C) = \frac{8}{36}$ and $P(A^c \cap B^c \cap C^c) = \frac{1}{36}$. Thus, we get the Karnaugh map shown on the right.





Hence

(a) the probability that at least one of the events A, B, and C occurs is $P(A \cup B \cup C) = 1 - P(A^c \cap B^c \cap C^c) = 1 - \dfrac{1}{36} = \dfrac{35}{36}$.

(b) the probability that at least two of the events A, B, and C occur is $P(AB \cap BC \cap AC) = \dfrac{12}{36} + \dfrac{4}{36} = \dfrac{16}{36} = \dfrac{4}{9}$.

(c) the probability that C did not occur given that exactly one of A and B^c occurred is $P(C^c|A \oplus B^c) = P(C^c|AB \cup A^cB^c) = P(C^c|A^cB^c) = \dfrac{P(A^cB^cC^c)}{P(A^cB^c)} = \dfrac{1/36}{9/36} = \dfrac{1}{9}$.

2. (a) If $\mathcal{X} = n$, the likelihood ratio has value

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1}{p_0}\left(\frac{1-p_1}{1-p_0}\right)^{n-1} > 1 \text{ if } (n-1)\ln\left(\frac{1-p_1}{1-p_0}\right) > \ln\left(\frac{p_0}{p_1}\right).$$

Since $p_1 < p_0$, we have that $1-p_1 > 1-p_0$ and $\ln((1-p_1)/(1-p_0)) > 0$. Therefore, the maximum likelihood decision rule is

$$\text{“Decide that } \mathsf{H}_1 \text{ is the true hypothesis if } \mathcal{X} > 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}\text{”}$$

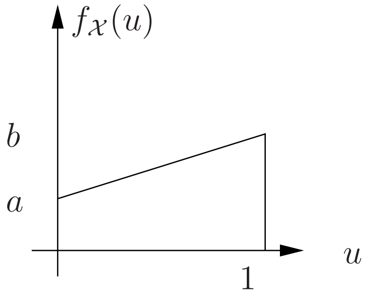
(b) The minimum-error-probability (MEP) decision rule decides that H_1 is the true hypothesis if the likelihood ratio exceeds the threshold π_0/π_1. Now $\Lambda(1) = p_1/p_0 < 1$. Since $1 - p_1 > 1 - p_0$, we see that

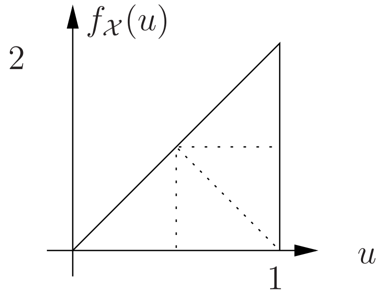
$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1(1-p_1)^{n-2}}{p_0(1-p_0)^{n-2}}\left(\frac{1-p_1}{1-p_0}\right) = \Lambda(n-1)\left(\frac{1-p_1}{1-p_0}\right) > \Lambda(n-1),$$

and so $\Lambda(1) = p_1/p_0$ is the smallest value of the likelihood ratio. It follows that if $\pi_0/\pi_1 = \pi_0/(1-\pi_0) < p_1/p_0$, that is, if $\pi_0 < p_1/(p_0+p_1)$, the MEP decision rule will always decide that H_1 is the true hypothesis regardless of the observed value of $\mathcal{X}$.

On the other hand, since $\Lambda(n)$ increases monotonically without bound as n increases, there is *no* value of $\pi_0 < 1$ for which π_0/π_1 can be guaranteed to be larger than the likelihood ratio no matter what value $\mathcal{X}$ takes on.

3. (a) The pdf of $\mathcal{X}$ is as shown in the left-hand figure below.



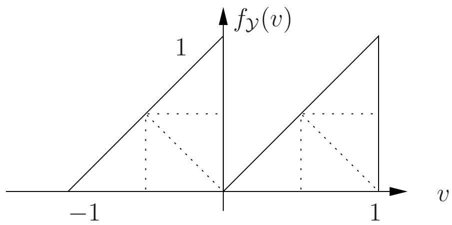


The trapezoidal figure has area $\frac{1}{2}$(sum of paralllel sides)$\times$(distance between parallel sides) $=$ $(a+b)/2$, and since the area must be 1, we conclude that $a+b=2$. Next, note that

$$\mathsf{E}[\mathcal{X}] = \frac{2}{3} = \int_0^1 u(a+(b-a)u)\,du = a\frac{u^2}{2} + (b-a)\frac{u^3}{3}\bigg|_0^1 = \frac{2b+a}{6}$$

which together with $a+b=2$ gives that $a=0, b=2$, and so the pdf is as shown in the right-hand figure above: $f_{\mathcal{X}}(u) = 2u$ for $0 \le u \le 1$. It follows almost by inspection that $P\left\{\mathcal{X} < \frac{1}{2}\right\} = \frac{1}{4}$.

(b) The pdf of $\mathcal{Y}$ is as shown below.



By inspection, $P\left\{|\mathcal{Y}| < \frac{1}{2}\right\} = \frac{1}{2}$ and $P\left\{\mathcal{Y} > 0 \,\middle|\, \mathcal{Y} < \frac{1}{2}\right\} = \frac{P\left\{0 < \mathcal{Y} < \frac{1}{2}\right\}}{P\left\{\mathcal{Y} < \frac{1}{2}\right\}} = \frac{1}{5}$.

Finally, $\mathsf{E}[\mathcal{Y}] = \int_{-1}^{0} v(1+v)\,dv + \int_0^1 v^2\,dv = \frac{v^2}{2} + \frac{v^3}{3}\bigg|_{-1}^{0} + \frac{v^3}{3}\bigg|_0^1 = \frac{-1}{6} + \frac{1}{3} = \frac{1}{6}$.

Even this can be obtained without integration. As far as moments are concerned, the two triangles can be represented by masses of $1/2$ sitting at $2/3$ and $-1/3$ respectively (cf. part(a)), and so the center of mass is at the midpoint of the two locations: $(2/3 - 1/3)/2 = 1/6$.

4. $\mathcal{X}$ is a Gaussian random variable (mean 60, variance 400).

(a) $P\{\mathcal{X} < 0\} = \Phi\left(\frac{0-60}{20}\right) = \Phi(-3) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013$.

(b) $P\{\mathcal{X} < 32 \,|\, \mathcal{X} > 0\} = \frac{P(\{\mathcal{X} < 32\} \cap \{\mathcal{X} > 0\})}{P\{\mathcal{X} > 0\}} = \frac{P\{0 < \mathcal{X} < 32\}}{P\{\mathcal{X} > 0\}} = \frac{\Phi(-1.4) - \Phi(-3)}{1 - \Phi(-3)}$
$= \frac{\Phi(3) - \Phi(1.4)}{\Phi(3)} = \frac{0.9987 - 0.9192}{0.9987} = \frac{795}{9987}$.