University of Illinois Fall 2006

ECE 413: Hour Exam II

Monday November 13, 2006
7:00 p.m. — 8:00 p.m.
119 Materials Science Building

1. [**18 points; 6 points per part**] Let A, B, and C denote events such that
$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{4}, \quad P(C) = \frac{2}{3}, \quad \text{and} \quad P(B|C) = \frac{1}{6}.$$
Suppose that A and B are *mutually exclusive* events, and that A and C are *mutually independent* events. Find
   (a) the probability that at least one of the events A, B, and C occurs,
   (b) the probability that at least two of the events A, B, and C occur,
   (c) the probability that C did not occur given that exactly one of A and B^c occurred.

2. [**30 points**] A coin is tossed repeatedly (independent trials) until a Head is observed for the first time. $\mathcal{X}$ denotes the number of trials needed to observe the first Head. The two hypotheses are
   - H_1: $\mathcal{X} \sim \mathsf{Geometric}(p_1)$
   - H_0: $\mathcal{X} \sim \mathsf{Geometric}(p_0)$

   where $p_1 < p_0$.
   (a) [**12 points**] The maximum-likelihood decision rule can be stated as
   "Decide that H_1 is the true hypothesis if $\mathcal{X}$? $g(p_0, p_1)$"
   where ? is either $<$ or $>$, and $g(p_0, p_1)$ is a function that you are asked to find.
   (b) [**18 points**] Let π_0 and $\pi_1 = 1 - \pi_0$ respectively denote the *a priori* probabilities of hypotheses H_0 and H_1 and assume that $0 < \pi_0 < 1$.
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation $\mathcal{X}$?
   For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_0 regardless of the value of the observation $\mathcal{X}$?

3. [**36 points**]
   (a) [**12 points**] $\mathcal{X}$ denotes a random variable with probability density function
$$f_{\mathcal{X}}(u) = \begin{cases} a + (b-a)u, & 0 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Given that $\mathsf{E}[\mathcal{X}] = \frac{2}{3}$, what is $P\left\{\mathcal{X} < \frac{1}{2}\right\}$?
   (b) [**24 points**] $\mathcal{Y}$ denotes a random variable with probability density function
$$f_{\mathcal{Y}}(v) = \begin{cases} 1 + v, & -1 \leq v \leq 0, \\ v, & 0 < v \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Find $P\left\{|\mathcal{Y}| < \frac{1}{2}\right\}$, $P\left\{\mathcal{Y} > 0 \,\middle|\, \mathcal{Y} < \frac{1}{2}\right\}$, and $\mathsf{E}[\mathcal{Y}]$.

4. [**16 points**] $\mathcal{X}$ is a Gaussian random variable (mean 60, variance 400) that models the average daily temperature (in °F) in a certain city.
   (a) What is the probability that the temperature is below 0°F?
   (b) What is the probability that the temperature is below freezing (32°F) given that it is above 0°F? You may leave your answer as the ratio of two integers