University of Illinois Fall 2006

ECE 413: Hour Exam I

Monday October 9, 2006
7:00 p.m. — 8:00 p.m.
119 Materials Science Building

1. [20 points] Let A, B, and C denote three events defined on a sample space Ω, and suppose that $P(A) = P(B) = 0.4, P(C) = 0.5$, and $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$.

   Find the following probabilities: $P(A \cap B)$, $P(A^c \cap B)$, $P((A \cup B \cup C)^c)$, $P(C^c | (A^c \cap B^c))$

2. [24 points] A bag contains $n \geq 2$ pairs of shoes in distinct styles and sizes.

   (a) You pick two shoes at random from the bag. Note that this is sampling without replacement.

   i. [5 points] What is the probability that you get a pair of shoes?
   ii. [5 points] What is the probability of getting one left shoe and one right shoe?

   (b) You now pick a third shoe at random from the bag *without* returning the two shoes that you previously picked to the bag.

   i. [7 points]What is the probability that you have a pair of shoes among the three that you have picked?
   ii. [7 points] What is the probability that there is at least one left shoe and at least one right shoe among the three?

3. [36 points] Fred and Wilma take turns tossing a coin with $P(\text{Heads}) = p$ and $P(\text{Tails}) = q = 1 - p$ where $0 < p, q < 1$. Fred tosses first, then Wilma, then Fred again, and so on until the game ends. Let F_i and W_i respectively denote the events that Fred and Wilma win the i-th game.

   (a) The rules are that the first one to toss a Head wins the game. In succeeding games, the *loser* of the previous game tosses first. Remember that Fred tosses first in the first game.

   (b) [6 points] Find $P(F_1)$. Which is larger: $P(F_1)$ or $P(W_1)$?

   (c) [10 points] Remember that the *loser* of the first game tosses first in the second game. What is the probability that Fred wins the second game? Which is larger, $P(F_2)$ or $P(W_2)$?

   In a different game, the rule is that the first one to *match* the previous toss wins the game. Wilma graciously insists that Fred go first as usual in the first game, and the schmuck accepts without realizing that he has no previous toss to match! Note that there are at least two coin tosses in the first game, and that HH or TT constitute a win for Wilma, while HTT and THH constitute a win for Fred, etc.

   (a) [10 points] Find $P(F_1)$. Which is larger: $P(F_1)$ or $P(W_1)$?

   (b) [10 points] The *first toss* of the second game is the *winning* toss of the first game. Thus, outcomes HHH or TTT constitute a win for Wilma (her first toss matches Fred's first toss) in the first game *and* a win for Fred in the second game (his second toss matches Wilma's previous winning toss). What is the probability that Fred wins the second game? Which is larger, $P(F_2)$ or $P(W_2)$?

4. [20 points]

   (a) [6 points] What is $\mathsf{E}[\mathcal{X}^2]$ for a Poisson random variable $\mathcal{X}$ with mean 5 ?

   (b) [6 points] If $\mathcal{Y}$ is a geometric random variable with mean 2, what is $\mathsf{var}(2 - 3\mathcal{Y})$?

   (c) [8 points] If $\mathcal{Z}$ denotes the number of occurrences of an event of probability p on 10 independent trials, what is the *conditional* expected value of $\mathcal{Z}$ given that the event occurred 4 times on the first six trials?