UNIVERSITY OF ILLINOIS, URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 313: Probability with Engineering Applications-Fall 2003

First Midterm Exam Solution

Problem 1 (*18 points*) Let A and B be arbitrary events with $0 < P(A), P(B) < 1$. Indicate whether each of the following statements is true or false by clearly writing "true" or "false" on the line provided after each letter.

(a) ______: $P(A \cup B) \leq P(A) + P(B)$.

(b) ______: $P(A|B) \leq 1$.

(c) ______: $P(A|B) + P(A|B^c) = 1$.

(d) ______: $P(A|B) + P(A^c|B) = 1$.

(e) ______: $\frac{P(A|B)}{P(B)} \leq 1$.

(f) ______: $P(AB^c) = P(A) - P(AB)$.

True, True, False, True, False, True, respectively.

Problem 2 (*12 points*) X is a Binomial $(10, p)$ random variable. Compute the following:

(a) The probability that X is equal to two. **(b)** The variance of $2 + 3X$.

(a) $\binom{10}{2} p^2(1-p)^8 = \frac{10 \cdot 9}{2} p^2(1-p)^8 = 45p^2(1-p)^8$.

(b) $9Var(X) = 90Var(Z_1) = 90p(1-p)$, where Z_i is a Bernoulli random variable.

Problem 3 (*5 points*) X is a random variable with mean and variance equal to unity: $\mu_X = \sigma_X^2 = 1$.

Compute $E[X^2]$.

$E[X^2] = E^2[X] + Var(X) = 2$.

Problem 4 (*20 points*) The random variables X and Y are independent, and each is uniformly distributed on the four integers $\{1, 2, 3, 4\}$. Compute the following:

(a) $E[X]$, **(b)** $E[XY]$, **(c)** $P(X = k \mid XY = 4)$, $k = 1, 2, 3, 4$.

(a) $E[X] = \frac{1+2+3+4}{4} = 2.5$.

(b) $E[XY] = E[X]E[Y] = (2.5)^2$.

(c) For $k = 1, 2, 4$, $P(X = k|XY = 4) = \frac{P(XY=4|X=k)}{P(XY=4)} \cdot P(X = k) = \frac{\frac{1}{16}}{\frac{3}{16}} = \frac{1}{3}$. For $k = 3$, $P(X = k|XY = 4) = 0$.

...Over

Problem 5 (*20 points*) A pair of dice is rolled until a sum of either 5 or 7 appears.

(a) Let E_n denote the event that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $n-1$ rolls. Compute $P(E_n)$.

(b) Compute the probability that a 5 occurs first.

(c) Compute the probability that a 7 occurs first.

(a) Let (X_i, Y_i) be the number on the ith roll of each dice. Then, $P(X_i + Y_i = 5) = \frac{4}{36} = \frac{1}{9}$ and $P(X_i + Y_i = 7) = \frac{6}{36} = \frac{1}{6}$. Also, $P(X_i + Y_i = 5 \text{ or } 7) = \frac{10}{36} = \frac{5}{18}$. Finally, $P(X_n + Y_n = 5 \text{ and } X_i + Y_i \neq 5 \text{ or } 7, 1 \leq i \leq n-1) = (\frac{13}{18})^{n-1}\frac{1}{9}$.

(b) $P(5 \text{ occurs first}) = \frac{1}{9}\sum_{n=1}^{\infty}(\frac{13}{18})^{n-1} = \frac{2}{5}$.

(c) $P(7 \text{ occurs first}) = \frac{1}{6}\sum_{n=1}^{\infty}(\frac{13}{18})^{n-1} = \frac{3}{5}$.

Problem 6 (*25 points*) A consumer electronics company purchases one million resistors from three different suppliers:

20% purchased from Supplier A, 30% from Supplier B, 50% from Supplier C

Of the resistors purchased, a certain number are defective, depending upon the supplier:

5% from Supplier A, 1% from Supplier B, 2% from Supplier C

A single resistor among these million is selected at random.

(a) Compute $P(D \mid B)$ = the probability the resistor is defective, given it came from Supplier B.

(b) Compute $P(D)$ = the probability the resistor is defective.

(c) Compute $P(B \mid D)$ = the probability the resistor came from Supplier B, given that it is defective.

(d) It is found that the resistor *is defective*. Express the maximum likelihood estimate of the supplier that sold this resistor, based on the quantities $P(D \mid B)$, $P(A \mid D)$, etc. You do not have to compute these conditional probabilities.

(e) Express the MAP estimate of the supplier that sold the faulty resistor, in terms of the same quantities as in (d).

(a) This is given as $P(D|B) = 0.01$.

(b) $P(D) = P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C) = (0.2)(0.05) + (0.3)(0.01) + (0.5)(0.02) = 0.023$.

(c) $P(B|D) = P(D|B) \cdot \frac{P(B)}{P(D)} = \frac{(0.01)(0.3)}{0.023} = \frac{3}{23}$.

(d) $\hat{\theta} = A$ if $P(D|A) > \max\{P(D|B), P(D|C)\}$. In fact, $\hat{\theta} = C$ if the resistor is defect. If the resistor is *not* defective, $\hat{\theta} = A$.

(e) $\hat{\theta} = A$ if $P(A|D) > \max\{P(B|D), P(C|D)\}$.

Useful fact: $1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x| < 1$.