UNIVERSITY OF ILLINOIS, URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

**ECE 313: Probability with Engineering Applications-Fall 2003**

SECOND MIDTERM EXAM SOLUTION

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**Name** Solution Key

**Section:** 10:00 AM 11:00 AM

**Score** 100

| Problem | Pts. | Score |
|---|---|---|
| 1 | 36 | |
| 2 | 4 | |
| 3 | 5 | |
| 4 | 5 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 13 | |
| 8 | 12 | |
| Total | 100 | |

**Please do not turn this page over until told to do so.**

You may not use any books, calculators, or notes other than two handwritten two-sided sheets of 8.5” x 11” paper.

**GOOD LUCK!**

**(36 Pts.)**

1. Answer TRUE or FALSE to each of the following questions:

   (a) If $X \sim U[0,1]$, then $X^2 \sim U[0,1]$. **T/F**
   False

   (b) If $X \sim N(0,1)$, then $Y = 2X + 1 \sim N(1,4)$. **T/F**
   True

   (c) Let $X$ be a uniform RV with mean $\mu$ and variance $\sigma^2$ and $Y$ be a Gaussian RV with the same mean and variance. Then $P(|X-\mu| > \sigma) = P(|Y-\mu| > \sigma)$. **T/F**
   False

   (d) Let $X$ be an exponential RV with parameter $\lambda$. Then $P(X > 5|X > 2) = e^{-3\lambda}$. **T/F**
   True

   (e) Let $X \sim N(0, \sigma^2)$ and $Y = X^2$. Then $E(Y) = \sigma^2$. **T/F**
   True

   (f) Let $F_X(u)$ be the CDF of a discrete RV $X$. If $F_X'(a) = 0$, then $P(X = a) \equiv 0$. **T/F**
   True

   (g) Let $(X,Y)$ be a discrete random vector taking on values $\{(u_i, v_j), i, j = 0, 1, \ldots\}$. Assume that the joint pmf of $X$ and $Y$ is $p_{X,Y}(u_i, v_j)$ and the pmf of $X$ is $p_X(u_i)$. Then $p_X(u_i) \geq p_{X,Y}(u_i, v_j)$ for all $v_j$. **T/F**
   True

   (h) Let $(X,Y)$ be a discrete random vector taking on values $\{(u_i, v_j), i, j = 0, 1, \ldots\}$. Assume that the joint pmf of $X$ and $Y$ is $p_{X,Y}(u_i, v_j)$ and the conditional pmf of $X$ given that $Y = v_j$ is $p_{X|Y}(u|v_j)$. Then $p_{X|Y}(u|v_j) \geq p_{X,Y}(u, v_j)$ for all values of $u$ and $v_j$. **T/F**
   True

   (i) Let $(X,Y)$ be a continuous random vector defined over the entire 2D plane. Assume that the joint pdf of $X$ and $Y$ is $f_{X,Y}(u,v)$ and the pdf of $X$ is $f_X(u)$. Then $f_X(u) \geq f_{X,Y}(u,v)$ for all values of $v$. **T/F**
   False

   (j) Let $(X,Y)$ be a continuous random vector defined over the entire 2D plane. Assume that the joint pdf of $X$ and $Y$ is $f_{X,Y}(u,v)$ and the conditional pdf of $X$ given that $Y = v_0$ with $f_Y(v_0) > 0$ is $f_{X|Y}(u|v_0)$. Then $f_{X|Y}(u|v_0) \geq f_{X,Y}(u, v_0)$ for all values of $u$ and $v_0$. **T/F**
   False

   (k) Let $X$ be a random variable characterizing the lifetime of a system with hazard rate function $\lambda(t)$. Then, $P(X > t) = \int_0^t \lambda(\tau) d\tau$. **T/F**
   False

   (l) Let $X$ and $Y$ be two random varibales related to each other by $Y = g(X)$ with $g(\cdot)$ being a **montonic** function. Then $P(Y > a) = P(X > g^{-1}(a))$ is always true. **T/F**
   False

**(4 Pts.)**

2. If $X \sim U[0,1]$ and $Y = 2X + 1$, then $Y \sim U[a,b]$ with $a$ and $b$ given below:

   (a) $[a,b] = [0,1]$
   (b) $[a,b] = [1,2]$
   (c) $[a,b] = [-1,-2]$
   (d) $[a,b] = [1,3]$
   (e) $[a,b] = [-1,-3]$
   (f) None of the above

   (d)

**(5 Pts.)**

3. Let $Y = |X|$, then

   (a) $E(Y) \geq E(X)$
   (b) $E(Y) \leq E(X)$
   (c) $E(Y) = |E(X)|$
   (d) None of the above

   (a)

**(5 Pts.)**

4. Let $X \sim U[a,b]$ has mean $\mu$ and variance $\sigma^2$. Determine $a$ and $b$.

   Since $X \sim U[a,b]$, $\mu = \frac{a+b}{2}$ and $\sigma^2 = \frac{(b-a)^2}{12}$. Also, since $\sigma > 0$, $\sigma = \frac{b-a}{2\sqrt{3}}$. Therefore, solving the following set of equations

$$a + b = 2\mu$$
$$b - a = 2\sqrt{3}\sigma,$$

   yields

$$a = \mu - \sqrt{3}\sigma, \text{ and } b = \mu + \sqrt{3}\sigma.$$

**(10 Pts.)**

5. Let $X \sim U[0,1]$ and $Y$ is an exponential RV with parameter $\lambda$. Assume that $Y = g(X)$. Determine $g(\cdot)$.

$$\begin{aligned} f_X(u) &= \lambda e^{-\lambda u}, \quad u \geq 0 \\ F_X(u) &= \int_0^u f_X(t)dt = 1 - e^{-\lambda u}, \quad u \geq 0 \\ Y = F_X^{-1}(x) &= -\frac{1}{\lambda}\ln(1-x). \end{aligned}$$

**(15 Pts.)**

6. Let $X_1, X_2, \dots, X_n$ be independent RVs with the same exponential distribution with parameter $\lambda$. Determine $P(\min\{X_1, X_2, \dots, X_n\} < a)$, for a constant $a > 0$.

$$\begin{aligned} P(\min\{X_1, X_2, \dots, X_n\} < a) &= 1 - P(\min\{X_1, X_2, \dots, X_n\} \geq a) \\ &= 1 - P(X_1 \geq a, X_2 \geq a, \dots, X_n \geq a) \\ &= 1 - \prod_{i=1}^{n} P(X_i \geq a) \\ &= 1 - e^{-\lambda n a}. \end{aligned}$$

**(13 Pts.)**

7. You are given $\Phi(u)$, the CDF of a standard Gaussian RV, only for $u \geq 0$. Let $X \sim N(1, 2^2)$. Determine the following probabilities using the given values of $\Phi(u)$.

   (a) $P(|X| < 3)$

   (b) $P(X^2 - 3X + 2 < 0)$

Note that

$$F_X(u) = \Phi\left(\frac{u-1}{2}\right)$$
$$\Phi(-u) = 1 - \Phi(u).$$

(a)

$$\begin{aligned} P(|X| < 3) &= P(-3 < X < 3) \\ &= F_X(3) - F_X(-3) \\ &= \Phi(1) - \Phi(-2) = \Phi(1) + \Phi(2) - 1. \end{aligned}$$

(b)

$$\begin{aligned} P(X^2 - 3X + 2 < 0) &= P((X-2)(X-1) < 0) \\ &= P(1 < X < 2) \\ &= F_X(2) - F_X(1) \\ &= \Phi\left(\frac{1}{2}\right) - \Phi(0). \end{aligned}$$

**(12 Pts.)**

8. The discrete random variables $X$ and $Y$ have joint pmf $p_{X,Y}(u,v)$ given by

| $v\downarrow \quad u\rightarrow$ | 0 | 1 | 3 | 5 |
|---|---|---|---|---|
| $-1$ | $\frac{1}{12}$ | $\frac{1}{6}$ | $\frac{1}{12}$ | 0 |
| 2 | $\frac{1}{6}$ | $\frac{1}{12}$ | 0 | $\frac{1}{12}$ |
| 4 | 0 | $\frac{1}{12}$ | $\frac{1}{6}$ | $\frac{1}{12}$ |

(a) Find the marginal pmf $p_X(u)$ and $p_Y(v)$.

(b) Calculate $P(1 < X + Y < 4)$.

(a) The marginal pmf $p_X(u)$ is

| $u$ | 0 | 1 | 3 | 5 |
|---|---|---|---|---|
| $p_X(u)$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{6}$ |

The marginal pmf $p_Y(v)$ is

| $v$ | $-1$ | 2 | 4 |
|---|---|---|---|
| $p_Y(v)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |

(b)

$$P(1 < X + Y < 4) = P(X = 0, Y = 2) + P(X = 1, Y = 2) + P(X = 3, Y = -1)$$
$$= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}.$$