**ECE 313** **Problem Set # 14** **Fall 2002**

**Assigned: 12/06/02** **Due: Friday 12/13/02**

Conditonal Expectation, Chebyshev's and Markov Inequalities
Weak Law of Large Numbers, and the Central Limit Theorem

**Assigned reading:** *Ross*, Sections 7.4.1-7.4.3, 7.5, 8-1-8.3
**Noncredit exercises:** Chapter 7, Problems 1,16, 26, 29, 34, 35 and Theoretical Exercises 1,2,17,22,23,40; Chapter 8, Problems 1-9, 15.
**Reminder: Final Exam:** Wednesday, December 18, 8:00-11 a.m., Room 138 Henry Administration Building. The exam is comprehensive with a slight emphasis on the material of the last four problem sets. You may bring two 8.5" by 11" sheets of notes to the hour exam. You may use both sides of the sheets, using font size 10 or larger (or similar handwriting size). The exam is closed book otherwise. Calculators, laptop computers, tables of integrals, etc. are not permitted.

**1. Correlation of histogram values**
Suppose that $n$ fair dice are independently rolled. Let

$$X_i = \begin{cases} 1 & \text{if a 1 shows on the } i^{th} \text{ roll} \\ 0 & \text{else} \end{cases} \qquad Y_i = \begin{cases} 1 & \text{if a 2 shows on the } i^{th} \text{ roll} \\ 0 & \text{else} \end{cases}$$

Let $X$ denote the sum of the $X_i$'s, which is simply the number of 1's rolled. Let $Y$ denote the sum of the $Y_i$'s, which is simply the number of 2's rolled. Note that if a histogram is made recording the number of occurrences of each of the six numbers, then $X$ and $Y$ are the heights of the first two entries in the histogram.
(a) Find $E[X_1]$ and $Var(X_1)$.
(b) Find $E[X]$ and $Var(X)$.
(c) Find $Cov(X_i, Y_j)$ if $1 \leq i, j \leq n$ (Hint: Does it make a difference if $i = j$?)
(d) Find $Cov(X, Y)$ and the correlation coefficient $\rho(X, Y)$ (called simply the correlation in *Ross*).

**2. Conditional expectation of histogram values**
Let the assumptions of the previous problem continue to hold.
(a) Find $E[Y|X = x]$ for any integer $x$ with $0 \leq x \leq n$. Note that your answer should depend on $x$ and $n$, but otherwise your answer is deterministic.
(b) Suppose that $Y$ is to be estimated from $X$. Find the function $g$ that minimizes the mean square error $E[(Y - g(X))^2]$.

**3. Covariance for a Poisson process**
Let $(N(t) : t \geq 0)$ be a Poisson process with rate $\lambda > 0$. Recall this means that
$N(t)$ counts the number of arrivals in the interval $[0, t]$;
the number of arrivals in an interval $[a, b]$ is a Poisson random variable with mean $\lambda(b - a)$; and the numbers of arrivals in disjoint intervals are independent.
Let $0 < \tau < T$ and let $n$ and $i$ be integers greater than or equal to zero.
(a) Find $E[N(\tau)|N(T) = n]$. Hint for (a) and (b): you may refer to a problem on a previous problem set.
(b) Find $E[N(T)|N(\tau) = i]$
(c) Find $Cov(N(\tau), N(\tau))$. (This is the same as $Var(N(\tau))$.)
(d) Find $Cov(N(\tau), N(T) - N(\tau))$.
(e) Find $Cov(N(\tau), N(T))$. (Hint: Use parts (c) and (d)).

**4. Conditional expectation for uniform density over a triangle**
Let $(X, Y)$ be uniformly distributed over the triangle with coordinates $(0, 0), (1, 0)$, and $(2, 1)$.
(a) What is the value of the joint pdf inside the triangle?
(b) Find the marginal density of $X$, $f_X(x)$. Be sure to specify your answer for all real values of $x$.
(c) Find the conditional density function $f_{Y|X}(y|x)$. Be sure to specify which values of $x$ the conditional density is well defined for, and for such $x$ specify the conditional density for all $y$. Also, for such $x$ briefly describe the conditional density of $y$ in words.
(d) Find the conditional expectation $E[Y|X = x]$. Be sure to specify which values of $x$ this conditional expectation is well defined for.

**5. Rock and roll**
A fair die is rolled until the number 1 appears. Let $N$ denote the number of rolls required (including the last roll when the 1 appears) and let $S = X_1 + X_2 + \ldots + X_N$.
(a) Find the pmf $p_N(n)$, the mean $E[N]$, and the variance $Var(N)$ on $N$. What type of random variable is $N$?
(b) Let $S$ denote the sum of the numbers rolled, including the "1" that appeared on the $N^{th}$ roll. Find $E[S|N = n]$ for any integer $n \geq 1$. (Hint: What is the conditional distribution of the outcomes observed for the first $n - 1$ rolls, given $N = n$?)
(c) Using your answer to part (b), find $E[S]$. Can you think of a simple reason $E[S] = 1 + 2 + 3 + 4 + 5 + 6$?

**6. Normal approximation for quantization error**
Suppose each of 100 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. Using the normal approximation suggested by the central limit theorem, find the probability that the absolute value of the sum of the errors is greater than 5.

**7. A comparison of bounds and the normal approximation**
A single fair die with the numbers 1 through 6 on it is rolled repeatedly. Let $X_i$ denote the number appearing on the $i^{th}$ roll. Assume the $X_i$'s are independent, and let $S_n = X_1 + X_2 + \ldots + X_n$.
(a) Use Markov's inequality to find an upper bound on $P[S_{100} \geq 400]$.
(b) Use Chebyshev's inequality to find an upper bound on $P[S_{100} \geq 400]$. (Hint: a one-sided tail probability can be bounded by a two-sided tail probability.)
(c) Compute the approximation to $P[S_{100} \geq 400]$ suggested by the central limit theorem.