ECE 313 **Problem Set # 12** **Fall 2002**
Assigned: 11/15/02 **Due: Wednesday 11/20/02**

More jointly distributed random variables

Assigned reading: *Ross*, Sections 6.1-6.5, 6.7, 5.4.1,and the first three pages of Chapter 7
Noncredit exercises: Chapter 6, Problems 1,10-15,20-23,28-30,41-43,51,54 and Theoretical Ex. 8,14,22,23,33

1. Uniform distribution on a rotated square
Random variables X and Y have a uniform joint density on the square bounded with corners at the points: $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$.
(a) Calculate the marginal pdfs of X and Y. Are X and Y independent?
(b) Compute $E[X]$ and $Var(X)$.
(c) Calculate the pdf of the random variable $A = X + Y$.
(d) Calculate the pdf of the random variable $C = X/Y$.

2. Functions of independent exponential random variables
Let X_1 and X_2 be independent random varibles, with X_i being exponentially distributed with parameter λ_i.
(a) Find the pdf of $Z = \min\{X_1, X_2\}$.
(b) Find the pdf of $R = \frac{X_1}{X_2}$.

3. Sums of independent normal random variables
Ross, Problem 33, page 293.

4. Sizing a Confidence Interval by the normal approximation to the binomial distribution
A campus network engineer would like to estimate the fraction p of packets going over the fiber optic link from the campus network to Bardeen Hall that are digital video disk (DVD) packets. The engineer writes a script to examine 1000 packets, counts the number X that are DVD packets, and uses $\hat{p} = \frac{X}{1000}$ to estimate p. The inspected packets are separated by hundreds of other packets, so it is reasonable to assume that each packet is a DVD packet with probability p, independently of the other packets.
(a) If $p = 0.5$, estimate $P[|\hat{p} - p| \geq 0.02]$.
(b) If $p = 0.1$, estimate $P[|\hat{p} - p| \geq 0.02]$
(c) If $p = 0.5$, find the number δ so that $P[|\hat{p} - p| \leq \delta] \approx 0.99$. Equivalently, $P[p \in [\hat{p} - \delta, \hat{p} + \delta]] \approx 0.99$. Note that p is not random, but the *confidence interval* $[\hat{p} - \delta, \hat{p} + \delta]$ is random.
(d) If $p = 0.1$, find the number δ so that $P[|\hat{p} - p| \leq \delta] \approx 0.99$.
(e) However, the campus network engineer doesn't know p to begin with, so she can't select the halfwidth δ of the confidence interval as a function of p. A reasonable approach is to select δ so that, the normal approximation to $P[p \in [\hat{p} - \delta, \hat{p} + \delta]]$ is greater than equal to 0.99 for any value of p. What is that value of δ?
(f) Using the same approch as in part (e), how many observations are needed (not depending on p) so that the (random) confidence interval $[\hat{p} - 0.01, \hat{p} + 0.01]$ contains p with probability at least 0.99 (according to the normal approximation of the binomial)?

5. Transformation of jointly continuous random variables, I
Suppose Z is exponentially distributed with parameter $\lambda = 1$, Θ is uniform on $(0, 2\pi)$, and Z is independent of Θ. Let $X = \sqrt{2Z}\cos\Theta$ and $Y = \sqrt{2Z}\sin\Theta$.
(a) Find the joint pdf of X and Y. In particular, show that X and Y are independent of each other, and each has the standard normal distribution.
(b) Find the pdf of the random variable $R = \sqrt{X^2 + Y^2}$. What is the name of this distribution?

6. Transformation of jointly continuous random variables, II
Suppose (U, V) has joint pdf

$$f_{U,V}(u, v) = \begin{cases} 9u^2v^2 & \text{if } \quad 0 \leq u \leq 1 \ \& \ 0 \leq v \leq 1 \\ 0 & \text{else} \end{cases}$$

Let $X = 3U$ and $Y = UV$. Find the joint pdf of X and Y, being sure to specify where the joint pdf is zero.
(b) Using the joint pdf of X and Y, find the conditional pdf, $f_{Y|X}(y|x)$, of Y given X. (Be sure to indicate

which values of x the conditional pdf is well defined for, and for each such x specify the conditional pdf for all real values of y.)