ECE 313 **Problem Set # 7** **Fall 2002**
Assigned: 10/9/16 **Due: Wednesday 10/16/02**

More discrete random variables

Assigned reading: Review *Ross*, Chapter 4.
Noncredit exercises: Chapter 4, Problems 35-43, 47-59, 70, and Theoretical Exercises 16-18, 25, 26.

1. Maximum variance Bernoulli and binomial random variables
(a) Suppose X is a Bernoulli random variable with parameter p. So $0 \le p \le 1$ and $P[X = 1] = p$ and $P[X = 0] = 1 - p$. What value of p maximizes the variance of X, and what is the maximum value of the variance?
(b) Suppose Y is a binomial random variable with parameters n and p. What value of p maximizes the variance of Y (for n fixed), and what is the maximum value of the variance?

2. Poisson distribution potpourri
(a) Let X be the number of leaves that fall off a certain tree in a one-minute interval of a sunny, windless autumn afternoon in Urbana-Champaign. Explain why X can be viewed as a binomial random variable with a very large parameter n and a very small parameter p. What is n? What is p?
(b) Accordingly, it is reasonable to suppose that X is a Poisson random variable with mean λ for some $\lambda > 0$. Suppose that λ is unknown, but it is decided that λ will be estimated by observing X. If $X = 12$ is observed, find the maximum likelihood estimate $\hat{\lambda}_{ML}$. This is the value of λ that maximizes $P[X = 12]$, or equivalently, maximizes $\ln P[X = 12]$.
(c) Express $P[X = 2|X \le 3]$ as a function of λ.

3. On the shape of the Poisson distribution
State and prove a version of Proposition 6.1 (*Ross*, Section 4.6.1) in case X is a Poisson random variable with parameter $\lambda > 0$, and k goes from 0 to ∞. (Hint: Very few changes are needed.)(Note: Working with the ratios $P[X = k]/P[X = k-1]$ is useful for numerical computation of binomial and Poisson distributions. See Sections 4.6.2 and 4.7.1.)

4. Poisson approximation and the birthday paradox
This problem is about the birthdays of the students in a class. For the sake of this problem, assume every year has 365 days (i.e. ignore leap years). Assume that the 365 possible birthdays are equally likely for each student, and that the birthdays of the different students are independent.
(a) Suppose the students reveal their birthdays one at a time. Given that the first k students have distinct birthdays, what is the probability that the $k + 1^{st}$ student has a birthday distinct from the first k?
(b) Find a fairly simple expression for the probability that the birthdays of the first k students are distinct. Compute your answer for $k = 10, 20$, and 30. (Hint: Use part (a). A calculator or simple computer program is useful. Your answers may be smaller than you might have guessed ahead of time. That is the "birthday paradox." Like most paradoxes, this one can be explained by sharpened intuition. Such explanation is the purpose of the remainder of this problem.)
(c) Consider the first k students. Let E_{ij} denote the event that the birthdays of students i and j are the same, for $1 \le i < j \le k$. Find $P[E_{12}]$, $P[E_{12}E_{13}]$, and $P[E_{12}E_{13}E_{23}]$. Are the events E_{ij} pairwise independent? Are they independent? (Hint: just check the definitions. Show your work.)
(d) Again consider the first k students. Let X denote the number of the events E_{ij} with $1 \le i < j \le k$ that are true. Note that $P[X = 0] = P[\text{all } k \text{ birthdays are distinct}]$. Intuitively, the events E_{ij} are approximately independent. There are $N = \binom{k}{2}$ such events, so that X has approximately the binomial distribution with parameters N and $p = \frac{1}{365}$. Since N is reasonably large (if k is large enough) and p is small, the random variable X has approximately the Poisson distribution with mean $\lambda = Np$. Using this observation, give a simple approximation to $P[X = 0]$ in terms of k and λ. Compare the numerical values to those found in part (b) for $k = 10$,$k = 20$, and $k = 30$. (Note: your answers are so small because λ is roughly proportional to k^2, not just proportional to k.)

5. Conditional lifetimes and the memoryless property of the geometric distribution

(a) Let X represent the lifetime, rounded up to an integer number of years, of a certain car battery. Suppose that the pmf of X is given by $p(k) = 0.2$ if $3 \le k \le 7$ and $p(k) = 0$ otherwise. Find the probability, $P[X > 3]$, that a three year old battery is still working. Given that the battery is still working after five years, what is the conditional probability that the battery will still be working three years later? (i.e. what is $P[X > 8|X > 5]$?)

(b) A certain Illini basketball player shoots the ball repeatedly from half court during practice. Each time, she makes a basket with probability p. Let Y denote the number of shots required for her to make the first basket. Express the probability that she needs more than three attempts to make a basket, $P[Y > 3]$, in terms of p. Given that she already missed the first five attempts, what is the conditional probability that she will need more than three additional attempts to make a basket? (i.e. what is $P[Y > 8|Y > 5]$)?

(c) What type of probability distribution does Y have?

6. Packet length choice for noisy channels

Realistically, communication links such as wireless communication links aren't totally predictable. Things such as latency and the fraction of bits that are correctly received vary with time, network load and other variables. A good approach to model such variability is to use probability theory. In the following we explore how the probability of byte error might affect the packet size you use for an application requiring reliable transfer of a large amount of data. Use the following assumptions. Suppose that each packet consists of $h+n$ bytes, where a byte is eight bits, such as 00100011. The first h bytes of a packet comprise the header, and the remaining n bytes comprise the data bytes. The header contains information such as the destination address, a sequence number, some parity bytes, and so on. Suppose that the probability a byte is received in error is $p_{e,Byte}$, and that the error events for different bytes are independent. Assume that a packet is successfully received if all $n + k$ bytes are received correctly, and that a packet is detected to be in error otherwise. (Packets with errors have to be retransmitted.) (a) Express the probability a packet is successfully received, $p_{S,packet}$, in terms of n, h and $p_{e,Byte}$.

(b) Let X denote the number of data bytes successfully received as the result of a single packet transmission. Possible values of X are 0 or n. The efficiency η is defined to be $E[X]$ divided by $n+h$, which is the number of data bytes successfully received per byte transmitted. Find η as a function of n, h, and $p_{e,Byte}$.

(c) Why is η small if n is very small, and h is fixed?

(d) Why is η small if n is very large, and h is fixed?

(e) Assuming that $h = 60$ bytes and $p_{e,Byte} = 0.01$ (a fairly noisy channel), find the value of the positive integer n to maximize η and find the resulting value of η.

7. Cumulative distribution functions

Let X be a random variable with the cumulative distribution function shown. Compute the following probabilities (a) $P[X \le 1]$, (b) $P[X > 1]$, (c) $P[|X| \le 3]$, and (d) $P[X^2 \le 9]$.