**ECE 313** | **Problem Set # 4** | **Fall 2002**
**Assigned: 9/18/02** | | **Due: 9/25/02**

# Bayes' formula and independence

**Assigned reading:** Finish reading *Ross* Chapter 3
**Noncredit exercises:** Chapter 3, problems 53,58,59,62,63,70-74,78,81

**1. Independent vs. mutually exclusive**
Two fair dice are rolled, and the 36 outcomes are equally likely. Define the events:

A="sum is 7"

B="both numbers are less than or equal to 4"

C="numbers are equal"

D="difference between numbers is 1"

(a) List all choices of two of these four events such that the two events are independent. (Show your work, as usual.)
(b) List all choices of two of these four events such that the two events are mutually exclusive (i.e. disjoint).

**2. Sequence of events coming true**
Ross, problem 18, p. 105.

**3. A posteriori probability of a cause**
Ross, problem 11, p. 104.

**4. Who voted?**
Ross, problem 14, p. 105.

**5. Independent vs. mutually exclusive, II**
(a) Suppose that an event $E$ is independent of itself. Show that either $P(E) = 0$ or $P(E) = 1$.
(b) Events $A$ and $B$ have probabilities $P(A) = 0.3$ and $P(B) = 0.4$. What is $P(A \cup B)$ if $A$ and $B$ are independent? What is $P(A \cup B)$ if $A$ and $B$ are mutually exclusive?
(c) Now suppose that $P(A) = 0.6$ and $P(B) = 0.8$. In this case, could the events $A$ and $B$ be independent? Could they be mutually exclusive?

**6. Bayes' formula – your turn**
(a) Clearly state Bayes' formula for $P[F_i|E]$ for $1 \leq i \leq n$. Give an expression in which the denominator is a sum. Begin by including the assumptions about the events $F_1, F_2, \ldots, F_n$ and $E$.
(b) Give an *original* example of how Bayes' formula can be applied for the case $n = 3$. You are to think up the example yourself–it is not to be from any book or course notes, nor should it be the same as any other student's example. Identify the sample space $S$, the events $F_1$,$F_2$,$F_3$, and $E$, and numerical values for each term in each side of Bayes' formula. (Hint: Numerous examples are given in the text and in class. Try thinking of something related to another course you've taken, your favorite hobby, sport, current events, etc. The set of all outcomes must be partitioned into three events $F_1$,$F_2$ and $F_3$: e.g. the pizza in the box is small, medium or large. "Partitioned" means the three events are mutually exclusive and their union is $S$. Then there must be an observed event $E$, e.g. the pizza box is large. Typically observing $E$ gives some information about which of the $F_i$ is true. To complete your example specify $S$, the prior probabilities $P[F_i]$, and the conditional probabilities $P[E|F_i]$, and plug them into Bayes' formula. To continue our example, we take $S$ to be the set of nine ordered pairs, with the first entry of each pair giving the size of the pizza (S,M, or L) and the second entry giving the size of the box (S,M or L). For priori probabilities we can assume that, a priori, the pizza is equally likely to be small, medium, or large. As for conditional probabilities we can suppose that small pizzas are put into medium boxes, and medium and large pizzas are put into large boxes (so $P[E|F_1] = 0, P[E|F_2] = P[E|F_3] = 1$). Finally we plug the numbers into the formula, yielding

$$P[\text{pizza is large}|\text{box is large}] = \frac{1 \cdot \frac{1}{3}}{0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}} = \frac{1}{2}.$$

Be creative!)

**7. Single sensor**
A motion detector is used to detect the presence of a person in a room, as part of an energy saving temperature control system. The sensor outputs a value $Z$ with possible values $\{0, 1, 2, 3, 4\}$, with larger numbers tending to indicate that a person is present. Let $H_0$ denote the event (or hypothesis) that a person is absent and let $H_1$ denote the event that a person is present. The *likelihood matrix* for $Z$, giving the probabilities for possible outputs given $H_0$ or $H_1$, is as follows:

| | $Z = 0$ | $Z = 1$ | $Z = 2$ | $Z = 3$ | $Z = 4$ |
|---|---|---|---|---|---|
| $H_0$ | 0.80 | 0.08 | 0.06 | 0.04 | 0.02 |
| $H_1$ | 0.01 | 0.02 | 0.07 | 0.10 | 0.80 |

For example, $P[Z = 3|H_1] = 0.10$.
(a) Indicate for each of the five possible observation values the maximum likelihood (ML) decision of which hypothesis is true. This can be done by copying the likelihood matrix and simply circling the larger number in each column. Also, compute the value of the *likelihood ratio* $\Lambda(i) = \frac{P[Z=i|H_1]}{P[Z=i|H_0]}$ for $1 \leq i \leq 5$. The *likelihood ratio test* (LRT) with threshold $\tau$ is the decision rule that decides $H_1$ is true if $\Lambda(Z) > \tau$ and decides $H_0$ is true if $\Lambda(Z) < \tau$. (If $\Lambda(Z) = \tau$ then the decision can be made randomly.) Verify that the ML decision rule for this problem is equivalent to the LRT with threshold $\tau_{ML} = 1$.
(b) For the ML decision rule, compute the false alarm (conditional) probability and miss (conditional) probability defined by

$$\begin{aligned} p_{false\_alarm} &= P[decide\ H_1\ is\ true|H_0] \\ p_{miss} &= P[decide\ H_0\ is\ true|H_1]. \end{aligned}$$

For example, $p_{false\_alarm}$ is the sum of all the entries in the first row of the likelihood matrix that were *not* circled in part (a).
(c) Suppose it is decided on the basis of observations of occupancy patterns that $\pi_0 = P[H_0] = 0.8$ and $\pi_1 = P[H_1] = 0.2$. These probabilities $\pi_0$ and $\pi_1$ are called prior probabilities, because they are assumed to hold before the detector outputs are taken into account. Compute the *joint probability matrix*, which specifies $P[H_k, Z = i]$ for each possible hypothesis $H_k$ and for each possible observation value $i$. (The 10 numbers in the matrix sum to one.) Also, indicate for each possible observation value, the maximum a posteriori probability (MAP) decision of which hypothesis is true by circling the larger element of each column of the joint probability matrix. Verify that the MAP decision rule for this problem is equivalent to the LRT with threshold value $\tau_{MAP} = \frac{\pi_0}{\pi_1}$.
(d) For the MAP decision rule, compute $p_{false\_alarm}$, $p_{miss}$, and the average probability of error $p_{ave} = \pi_0 p_{false\_alarm} + \pi_1 p_{miss}$, using the same prior probabilities as in part (c). (Hint: The conditional probability $p_{false\_alarm}$ for the MAP decision rule is the sum of all the entries in the first row of the *likelihood matrix* that correspond to entries *not* circled in the first row of the *joint probability matrix*. The conditional probability $p_{miss}$ is computed similarly, using the second rows. As a check on your answer, $p_{ave}$ should equal the sum of the probabilities in the joint probability matrix that are not underlined. )
(e) Compute, for the same priors given in (c), the average error probability for the ML rule. Just use $p_{ave} = \pi_0 p_{false\_alarm} + \pi_1 p_{miss}$ with the $p_{false\_alarm}$ and $p_{miss}$ computed in part (b). Comment: Among all decision rules, the MAP rule minimizes $p_{ave}$, so your answer to (e) should be greater than or equal to your answer to part (d). On the other hand, the MAP rule requires knowledge of $\pi_0$ and $\pi_1$ whereas the ML rule does not.)