ECE 313: Probability with Engineering Applications

Fall 2002
Final Exam Solutions

Problem 1 T,T,F,T,F,T
Problem 2 T,F,T,T,F
Problem 3 F,T,F,F

Problem 4 (a) area=base $\times$ height=1, so density inside region is 1
(b)

$$f_X(x) = \begin{cases} \frac{x}{2} & : \quad 0 \le x \le 1 \\ \frac{1}{2} & : \quad 1 \le x \le 2 \\ \frac{3-x}{2} & : \quad 2 \le x \le 3 \\ 0 & : \quad \text{else} \end{cases}$$

(c) $E[X] = \frac{3}{2}$, $Var(X) = \frac{5}{12}$
(d) For $0 \le a \le 1$:

$$f_{Y|X}(y|a) = \begin{cases} \frac{2}{a} & : \quad 0 \le y \le \frac{a}{2} \\ 0 & : \quad \text{else} \end{cases}$$

(e) For $1 \le a \le 2$:

$$f_{Y|X}(y|a) = \begin{cases} 2 & : \quad \frac{a-1}{2} \le y \le \frac{a}{2} \\ 0 & : \quad \text{else} \end{cases}$$

(f)

$$E[Y_X = x] = \begin{cases} \frac{x}{4} & : \quad 0 \le x \le 1 \\ \frac{2x-1}{4} & : \quad 1 \le x \le 2 \\ \frac{x+1}{4} & : \quad 2 \le x \le 3 \\ 0 & : \quad \text{else} \end{cases}$$

(So slope is 1/4 over the intevals [0,1] and [2,3], and the slope is 1/2 over the interval [1,2].)

Problem 5 (a)

$$p_L(u) = \begin{cases} \frac{4}{7} & : \quad u = 1 \\ \frac{3}{7} & : \quad u = 0 \\ 0 & : \quad \text{else} \end{cases}$$

(b) The nonzero values of the pmf $p_{LR}(l, r)$ are shown in the table:

	$r = 0$	$r = 1$
$l = 0$	$\frac{6}{42}$	$\frac{12}{42}$
$l = 1$	$\frac{12}{42}$	$\frac{12}{42}$

(c) $Cov(L, R) = \frac{-2}{49}$
(d) No. One justification is that $Cov(L, R) \ne 0$. Another justification is that $P[L = 1, R = 1] \ne P[L = 1]P[R = 1]$.
(e)

$$p_Z(u) = \begin{cases} \frac{1}{7} & : \quad u = 0 \\ \frac{4}{7} & : \quad u = 1 \\ \frac{2}{7} & : \quad u = 2 \\ 0 & : \quad \text{else} \end{cases}$$

Problem 6 (Note that T has the exponential density with parameter 1/10.)
(a)

$$F_T(t) = \begin{cases} 1 - e^{-t/10} & : \quad t \geq 0 \\ 0 & : \quad t < 0 \end{cases}$$

(b) $E[T] = 10$
(c) $1 - e^{-1}$
(d) $\frac{1}{6e}$
(e)E[N]=10

Problem 7 (a) The ML rule can be expressed as a likelihood ratio test with threshold one. This leads to decide that H_1 is true if

$$\frac{\mu_1 e^{-\mu_1 X}}{\mu_0 e^{-\mu_0 X}} \geq 1$$

and decide H_0 is true otherwise. This simplifies to decide H_1 is true if

$$X \geq \frac{\ln(\mu_0/\mu_1)}{\mu_0 - \mu_1}$$

and decide H_0 is true otherwise.
(b)

$$p_{false\_alarm} = \left(\frac{\mu_1}{\mu_0}\right)^{\frac{\mu_0}{\mu_0 - \mu_1}}$$

$$p_{miss} = 1 - \left(\frac{\mu_1}{\mu_0}\right)^{\frac{\mu_1}{\mu_0 - \mu_1}}$$

(c) The MAP rule always decides that H_1 is true if and only if $\pi_0 \leq \frac{\mu_1}{\mu_0 + \mu_1}$.
(c) The MAP rule always decides that H_0 is true if and only if $\pi_0 = 1$.

Problem 8 (a) The best estimator of the form $aX + b$ is $E[Y] + \frac{Cov(X,Y)}{Var(X)}(X - E[X]) = 4 + \frac{25}{35}(X - 4)$. Equivalently, $a = \frac{8}{7}$ and $b = \frac{5}{7}$.
(b) The best estimator of the form cX is $\frac{E[XY]}{E[X^2]}X = \frac{41}{51}X$.
(c) The best estimator of the form cX is also the best estimator of the form $aX + b$ under the constraint that $b = 0$. Allowing $b \neq 0$ can only help. Thus, whenever the optimal b is not zero, the best estimator of the form $aX + b$ is better than the best estimator of the form cX.