# ECE 313: Probability with Engineering Applications
# Fall 2000, Exam I : Solutions

**Problem 1 Part (a) 5pts.** What is the probability that ball 5 will be among the three balls that are drawn.

$$\frac{\binom{5}{2}}{\binom{6}{3}} = \frac{10}{20} = 1/2$$

**Part (b) 5pts.**

We will use the general version of the theorem of total probability: $P(A) = \sum_i P(A|B_i)P(B_i)$, in which the $B_i$ partition the sample space. For this specific version of the problem, $A$ is the event that the chosen ball has an even number, and the $B_i$ are events corresponding to the possible ways to draw three balls from the first urn: $B_1$ is the event that the new urn contains three even numbered balls; $B_2$ is the event that the new urn contains two even numbered balls and one odd ball which is not the five ball; $B_3$ is the event that the new urn contains two even numbered balls and the five ball; $B_4$ is the event that the new urn contains one even numbered ball and two odd balls which are not the five ball; $B_5$ is the event that the new urn contains one even numbered ball and two odd balls, one of which is the five ball. The conditional probabilities are thus: $P(A|B_1) = 1$, $P(A|B_2) = 2/3$, $P(A|B_3) = 1$, $P(A|B_4) = 1/3$, $P(A|B_5) = 1/2$. Note that there is one additional $B_i$, (the event that there are no even balls in the the new urn) but since in this case the new urn would contain no even balls, $P(A|B_i) = 0$, and therefore it need not be included in the sum.

$$1 \times \frac{\binom{3}{3}}{\binom{6}{3}} + \frac{2}{3} \times \frac{\binom{3}{2}\binom{2}{1}}{\binom{6}{3}} + 1 \times \frac{\binom{3}{2}}{\binom{6}{3}} + \frac{1}{3} \times \frac{\binom{3}{1}\binom{2}{2}}{\binom{6}{3}} + \frac{1}{2} \times \frac{\binom{3}{1}\binom{2}{1}}{\binom{6}{3}} = 3/5$$

One could also make an argument by symmetry for this case. The five ball has been eliminated from consideration, but there is nothing in the description that singles out any of the remaining balls for special treatment. Therefore, the remaining five balls must be equally likely to be selected. If one could show that, e.g., the two ball were more likely to be chosen than the other four balls, then this same argument could be applied to any of the other non-five ball candidates, leading to a contradiction. Thus, since three of the five non-five ball candidates are even, the probability of drawing an even ball is 3/5.

**Part (c) 5pts.**

Again using the theorem of total probability, there are two cases to consider: the three ball and the five ball are both in the the new urn, the three ball is in the the new urn but the five ball is not. The conditional probability of the three ball being selected given that it is not in the the new urn is zero, and the corresponding terms in the summation can thus be ignored.

$$\frac{1}{2} \times \frac{\binom{4}{1}}{\binom{6}{3}} + \frac{1}{3} \times \frac{\binom{4}{2}}{\binom{6}{3}} = 1/5$$

One could also use the symmetry argument from part (b) to conclude that the three ball will be chosen with probability 1/5.

**Problem 2**

**Part (a) 5 pts.**

The easiest way to compute this probability is to count the ways in which $X_2$ is evenly divisible by $X_1$ and divide this by the total number of possible outcomes.
For $X_2 = 6$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1, 2, 3, 6\}$;
For $X_2 = 5$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1, 5\}$;
For $X_2 = 4$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1, 2, 4\}$;
For $X_2 = 3$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1, 3\}$;
For $X_2 = 2$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1, 2\}$;
For $X_2 = 1$, $X_2$ is evenly divisible by $X_1$ for $X_1 \in \{1\}$.
Since there are 36 possible outcomes, we have $p_Z(1) = \frac{14}{36} = \frac{7}{18}$ and $p_Z(0) = 1 - \frac{7}{18} = \frac{11}{18}$

**Part (b) 5 pts.**
If $X_2 = 4$ then $Z = 1$ when $X_1 \in \{1,2,4\}$, so $P(\{Z=1\} \mid \{X_2=4\}) = \frac{3}{6} = \frac{1}{2}$.
Alternatively, we could use

$$P(\{Z=1\} \mid \{X_2=4\}) = \frac{P(\{Z=1\},\{X_2=4\})}{P\{X_2=4\}} = \frac{\frac{3}{36}}{\frac{1}{6}} = \frac{3}{6} = \frac{1}{2}.$$

**Part (c) 5 pts.**
When $X_1 = 3$, $Z = 1$ if and only if $X_2 \in \{3,6\}$, so $P(\{Z=1\} \mid \{X_1=3\}) = \frac{2}{6} = \frac{1}{3}$.
Again, we could use

$$P(\{Z=1\} \mid \{X_1=3\}) = \frac{P(\{Z=1\},\{X_1=3\})}{P\{X_1=3\}} = \frac{\frac{2}{36}}{\frac{1}{6}} = \frac{2}{6} = \frac{1}{3}.$$

**Part (d) 5 pts.**
For this we can use Bayes theorem and our previous results:

$$P(\{X_1=3\} \mid \{Z=1\}) = \frac{P(\{Z=1\} \mid \{X_1=3\})P\{X_1=3\}}{P\{Z=1\}} = \frac{\frac{1}{3} \times \frac{1}{6}}{\frac{7}{18}} = \frac{1}{7}$$

**Problem 3** *(30 points)* It is believed by many professors that a student's grade in ECE 359 can be predicted based on whether or not the student passed ECE 313. In fact, in the faculty handbook, one can find the following likelihood matrix:

| Likelihood Matrix | | Grade in ECE 359 | | | | |
|---|---|---|---|---|---|---|
| | | A | B | C | D | F |
| $\mathbf{H}_0$: | Did not pass ECE 313 | 0 | 0 | 0.1 | 0.4 | 0.5 |
| $\mathbf{H}_1$: | Passed ECE 313 | 0.4 | 0.3 | 0.2 | 0.1 | 0 |

A professor who is teaching ECE 359 for the first time is trying to determine whether a particular student passed ECE 313 based on the student's grade in ECE 359.

**Part (a) 5 pts.**

| | Grade in ECE 359 | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | F |
| $\mathbf{H}_0$: | | | | X | X |
| $\mathbf{H}_1$: | X | X | X | | |

**Part (b) 5 pts.**

| Joint Prob Matrix | Grade in ECE 359 | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | F |
| $\mathbf{H}_0$: | 0 | 0 | 0.03 | 0.12 | 0.15 |
| $\mathbf{H}_1$: | 0.28 | 0.21 | 0.14 | 0.07 | 0 |

**Part (c) 5 pts.**
We can partition the sample space into ten disjoint events, which are shown below with their probabilities (taken directly from the joint probability matrix):
P { $\mathbf{H}_0$ is true and ECE 359 grade is an A } = 0.0
P { $\mathbf{H}_0$ is true and ECE 359 grade is a B } = 0.0
P { $\mathbf{H}_0$ is true and ECE 359 grade is a C } = 0.03
P { $\mathbf{H}_0$ is true and ECE 359 grade is a D } = 0.12
P { $\mathbf{H}_0$ is true and ECE 359 grade is an F } = 0.15
P { $\mathbf{H}_1$ is true and ECE 359 grade is an A } = 0.28
P { $\mathbf{H}_1$ is true and ECE 359 grade is a B } = 0.21
P { $\mathbf{H}_1$ is true and ECE 359 grade is a C } = 0.14
P { $\mathbf{H}_1$ is true and ECE 359 grade is a D } = 0.07
P { $\mathbf{H}_1$ is true and ECE 359 grade is an F } = 0.0

The probability of passing is simply the sum of the probabilities for the events where the grade is not an F: 0.85.

**Problem 3** *(cont)*

**Part (d) 5 pts.** Assuming the same prior probabilities as in Part (b), determine the the MAP decision rule and write it in the space below.

| | Grade in ECE 359 | | | | |
|---|---|---|---|---|---|
| | A | B | C | D | F |
| $\mathbf{H}_0$: | | | | X | X |
| $\mathbf{H}_1$: | X | X | X | | |

**Part (e) 5 pts.**

The probability of error can be computed by adding the probabilities associated with the events in which the wrong decsion is made. For the maximum likelihood rule, the wrong decision is made when:

$\mathbf{H}_0$ is true and ECE 359 grade is an A
$\mathbf{H}_0$ is true and ECE 359 grade is a B
$\mathbf{H}_0$ is true and ECE 359 grade is a C
$\mathbf{H}_1$ is true and ECE 359 grade is a D
$\mathbf{H}_1$ is true and ECE 359 grade is an F

By adding the corresponding probabilities (again taken directly from the joint probability matrix), we find that the probability of error is 0.1.

**Part (f) 5 pts.** Find the numbers $\pi_{min}$ and $\pi_{max}$ such that the ML and MAP decision rules are identical if and only if $\pi_{\min} \leq \pi_0 \leq \pi_{\max}$.

The ML and MAP rules agree for any values of the priors (i.e., $pi_0$ and $\pi_1$) that do not change the relative orderings on the elements in the columns when the joint probability matrix is constructed. It is easy to see that only two columns in this case could have the orderings changed due to multiplication by priors. In particular, the orderings will change when either $0.1\pi_0 > 0.2(1-\pi_0)$ or $0.4\pi_0 > 0.1(1-\pi_0)$. From here it's just algebra:

$$\begin{array}{ll} 0.1\pi_0 < 0.2(1-\pi_0) & 0.4\pi_0 > 0.1(1-\pi_0) \\ 0.1\pi_0 < 0.2 - 0.2\pi_0) & 0.4\pi_0 > 0.1 - 0.1\pi_0 \\ 0.3\pi_0 < 0.2 & 0.5\pi_0 > 0.1 \\ \pi_0 < \frac{2}{3} & \pi_0 > \frac{1}{5} \end{array}$$

And thus, $\pi_{min} = \frac{1}{5}$ and $\pi_{max}\frac{2}{3}$.

**Problem 4** *(15 points)*

**Part (a) 5 pts.**

Since there are two odd's and eight even's, the probability is $p^2(1-p)^8$.

**Part (b) 5 pts.**

The probability of observing $k$ odd rolls if the die is rolled $n$ times is equal to the probability of any specific outcome with $k$ odds and $(n-k)$ evens multiplied by the number of ways to have an outcome with $k$ odds and $(n-k)$ evens,

$$P\{k \text{ odd rolls}\} = \left( \begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

**Part (c) 5 pts.**

The probability of the above outcome is $p^2(1-p)^8$. The value for $\hat{p}_{ML}$ is the value of $p$ that maximizes this expression. It is easier to maximize the natural logarithm of this function:

$$\begin{aligned} 0 &= \frac{d}{dp} \ln \left[p^2(1-p)^8\right] \\ 0 &= \frac{d}{dp} \left[\ln p^2 + \ln(1-p)^8\right] \\ 0 &= \frac{d}{dp} \left[2 \ln p + 8 \ln(1-p)\right] \\ 0 &= \frac{2}{p} - \frac{8}{1-p} \\ 0 &= 2(1-p) - 8p \\ 0 &= 2 - 2p - 8p \\ 0 &= 2 - 10p \\ \hat{p}_{ML} &= 0.2 \end{aligned} \tag{1}$$