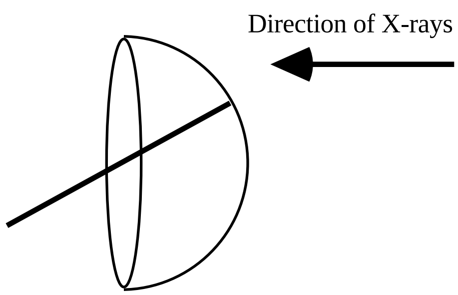
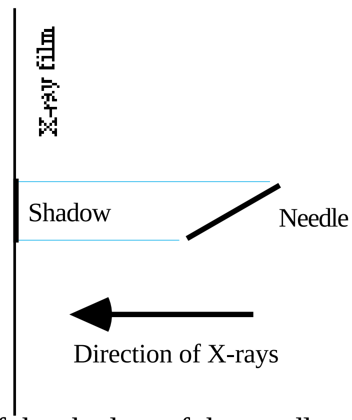
University of Illinois | **Problem Set #14** | **ECE 313 Fall 2001**

Assigned: Wednesday, November 26, 2001
Due: **Friday**, December 7, 2001
Reading: Ross, Chapter 7, Sections 1-5; Chapter 8 Sections 1-4
Noncredit Exercises: Chapter 7: Problems 1, 16, 26, 29, 34, 36;
Theoretical Exercises: 1, 2, 17, 22, 23, 40; Chapter 8: Problems 1-9, 15

Problems:

1. (Read Ross Example 2d on p. 255 (5th ed.) or p. 251 (6th ed.) first) Count Buffon showed that if a needle of length L is tossed onto a table on which parallel lines are ruled distance L apart, then the probability that the needle crosses one of the lines is $2/\pi$. By counting the number of times that the needle crossed a line in many actual trials of the experiment, he was able to determine π to a few decimal places. Unfortunately, one day, his assistant Buffoon swallowed the needle and had to be rushed to the hospital for X-rays. Assume that the needle is equally likely to be pointing in any direction in Buffoon's stomach, that is, the position of the tip of the needle that is closer to the source of the X-rays is uniformly distributed on the hemisphere (Buffoon's beer belly?) as shown.





(a) Find the probability density function of the length of the shadow of the needle on the X-ray film.

(b) Find the average length of the shadow of the needle on the X-ray film.

2. An VLSI chip has constant hazard rate $\lambda = -\ln 0.999$/week.

(a) What is the average lifetime (in weeks)? What is the median lifetime?

(b) What is the probability that the module lasts for at least one week?

Now suppose that three identical chips are organized into a triple–modular–redundancy (TMR) system in which we assume that the majority–logic gate cannot fail. Furthermore, we assume that the three modules fail independently of one another, that is, their lifetimes $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{X}_3$ are independent random variables. Thus, the three events $\{\mathbf{X}_1 > t_1\}$, $\{\mathbf{X}_2 > t_2\}$, and $\{\mathbf{X}_3 > t_3\}$ are independent for all possible values of t_1, t_2, t_3.
Let $\mathbf{Y}$ denote the length of time for which the TMR system functions correctly.

(c) If you are told that the event $\{\mathbf{Y} > t\}$ occurred, what can you say about the occurence (or nonoccurence) of the events $\{\mathbf{X}_1 > t\}$, $\{\mathbf{X}_2 > t\}$, and $\{\mathbf{X}_3 > t\}$?

(d) Show that $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$ and use this result to find the average lifetime and the median lifetime of the TMR system. Compare your answers to those in part (a). Do the results surprise you? Is the TMR system improving performance the way it is alleged to?

(e) What is the probability that the TMR system functions correctly for at least one week? Compare this answer to that of part (b). Do you think that the TMR system is more reliable or less reliable?

(f) Find t such that $P\{\mathbf{Y} > t\} = 0.999$ and compare the answer to that of part (b). Has the TMR system improved performance?

3. Let $E[\mathbf{X}] = 1$, $E[\mathbf{Y}] = 4$, $\text{var}(\mathbf{X}) = 4$, $\text{var}(\mathbf{Y}) = 9$, and $\rho_{\mathbf{X},\mathbf{Y}} = 0.1$

(a) If $\mathbf{Z} = 2(\mathbf{X}+\mathbf{Y})(\mathbf{X}-\mathbf{Y})$, what is $E[\mathbf{Z}]$?

(b) If $\mathbf{T} = 2\mathbf{X}+\mathbf{Y}$ and $\mathbf{U} = 2\mathbf{X}-\mathbf{Y}$, what is $\text{cov}(\mathbf{T}, \mathbf{U})$?

(c) If $\mathbf{W} = 3\mathbf{X} + \mathbf{Y} + 2$, find $E[\mathbf{W}]$ and $\text{var}(\mathbf{W})$.

(d) If $\mathbf{X}$ and $\mathbf{Y}$ are jointly Gaussian random variables, and $\mathbf{W}$ is as defined in (c), what is $P\{\mathbf{W} > 0\}$?

4. This problem has three independent parts. Do not apply the numbers from one part to the others.

(a) If $\text{var}(\mathbf{X} + \mathbf{Y}) = 36$ and $\text{var}(\mathbf{X} - \mathbf{Y}) = 64$, what is $\text{cov}(\mathbf{X}, \mathbf{Y})$? If you are also told that $\text{var}(\mathbf{X}) = 3 \bullet \text{var}(\mathbf{Y})$, what is $\rho_{\mathbf{X},\mathbf{Y}}$?

(b) If $\text{var}(\mathbf{X} + \mathbf{Y}) = \text{var}(\mathbf{X} - \mathbf{Y})$, are $\mathbf{X}$ and $\mathbf{Y}$ uncorrelated ?

(c) If $\text{var}(\mathbf{X}) = \text{var}(\mathbf{Y})$, are $\mathbf{X}$ and $\mathbf{Y}$ uncorrelated ?

5. Consider the random point $(\mathbf{X},\mathbf{Y})$ of Problem 2 of Problem Set #13.

(a) Compute $E[\mathbf{X}]$ and $\text{var}(\mathbf{X})$.

(b) Explain why the random variable $\mathbf{Y}$ has the same mean and variance as $\mathbf{X}$.

(c) Compute $E[\mathbf{XY}]$ and hence find $\text{cov}(\mathbf{X},\mathbf{Y})$.

(d) GLOTUS tells us that $E[g(\mathbf{X},\mathbf{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) \bullet f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$. Now, consider the function $g(\mathbf{X},\mathbf{Y}) = \min\{\mathbf{X},\mathbf{Y}\}$. Use GLOTUS to find $E[g(\mathbf{X},\mathbf{Y})] = E[\min\{\mathbf{X},\mathbf{Y}\}]$ by showing that the glotal integrand $g(u,v) \bullet f_{\mathbf{X},\mathbf{Y}}(u,v)$ can be expressed as

$u \bullet f_{\mathbf{X},\mathbf{Y}}(u,v)$ for all points (u,v) in the plane for which $u < v$

and as $v \bullet f_{\mathbf{X},\mathbf{Y}}(u,v)$ for all points (u,v) in the plane for which $v \le u$.

Thus, the glotal integral can be expressed as the sum of integrals over two disjoint regions (these are divided by the line of slope 1 through the origin) of the plane.

Reminder: In case you got lost in the above verbiage, you are to find $E[\min\{\mathbf{X},\mathbf{Y}\}]$.

(e) Repeat part (d) to find $E[h(\mathbf{X},\mathbf{Y})] = E[\max\{\mathbf{X},\mathbf{Y}\}]$.

(f) Compare the numerical values that you obtained in parts (d) and (e), and state whether or not $E[\max\{\mathbf{X},\mathbf{Y}\}]$ is larger than $E[\min\{\mathbf{X},\mathbf{Y}\}]$.

Should $E[\max\{\mathbf{X},\mathbf{Y}\}]$ exceed $E[\min\{\mathbf{X},\mathbf{Y}\}]$ (even if, according to your computed values, it does not in this instance)?

(g) Since $\min\{\mathbf{X},\mathbf{Y}\} + \max\{\mathbf{X},\mathbf{Y}\} = \mathbf{X} + \mathbf{Y}$, the following equation

$$E[\min\{\mathbf{X},\mathbf{Y}\}] + E[\max\{\mathbf{X},\mathbf{Y}\}] = E[\mathbf{X}] + E[\mathbf{Y}]$$

should hold. ***Is*** the above equation satisfied by the numerical values you obtained?

(h) The conditional pdf of $\mathbf{X}$ given $\mathbf{Y} = \alpha$ was obtained in Problem 2 of Problem Set #13, and it is easy to see that the conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = \alpha$ is similar. Now, the **best** (least mean-square error) estimate of $\mathbf{Y}$ given $\mathbf{X} = \alpha$ is the mean of the conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = \alpha$. Thus, if $\mathbf{X}$ has value $\alpha \le 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of $\mathbf{Y}$, is 0.75 while if $\mathbf{X}$ has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of $\mathbf{Y}$ (given that $\mathbf{X}$ is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b are given in Eq. (5.4) of Chapter 7 of Ross. Compute a and b, and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α. (Remember that $0 \le \alpha \le 1$). For what value(s) of α are the two estimates the same?

(i) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of $\mathbf{X}$, they really are *functions* of $\mathbf{X}$, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, & 0 \le \mathbf{X} \le 0.5, \\ 0.5, & 0.5 < \mathbf{X} \le 1 \end{cases}$ and $\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?